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WATER WAVES III.

by

John V. Wehausen

Under Contract Number N-onr-222(30)

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University of California
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May 1959

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INDEX

22. Initial-value problems

In the special problems considered in sections 17 to 21, the dependence upon the time has been precipitated out, either by assuming the motion steady in a moving coordinate system or by assuming a harmonic dependence upon the time. In this section we shall consider motions in which the displacement and velocity of the surface are specified at some instant of time, say $t = 0$, the motions of any solid boundaries are given for each instant $t \geqslant 0$ (except at the very end where freely floating bodies are considered) and the pressure distribution over the free surface is a given function for $t \geqslant 0$.

It is not usually possible in the most general situations to give explicit solutions for such problems. However, Volterra [1934] has proved a uniqueness theorem and has shown how to reduce the problem to finding an appropriate Green's function. His results were later rediscovered and extended to a wider class of problems by Finkelstein [1957] (see also Stoker [1957, ch. 6]). However, the use of Green's functions for initial-value problems extends back even earlier, at least to the papers of Hadamard [1910, 1916] and Bouligand [1913]. These theorems are discussed in section 22α.

One of the classical problems in this category is associated with the names Cauchy [1827] and Poisson [1815]. In this problem the fluid is infinite in horizontal extent, without obstructions, and either infinitely deep or of uniform depth h. At the initial instant $t = 0$, the form of the surface and its vertical velocity are given and one seeks the subsequent motion. Such problems have already been discussed at some length in section 15. However, in

that section interest centered upon investigation of certain aspects of the subsequent motion rather than upon obtaining the solution. In addition, the treatment of that section was limited to two-dimensional motion, although the methods could have been extended to three-dimensional motion.

The history of this problem, including an exposition of the methods used by various authors, is included in a paper by Risser [1924, pp. 113-144]. Another expository account can be found in Vergne [1928, ch. I]. The problem is discussed here in section 22β.

In section 22γ several special initial-value problems are discussed.

22α. Some general theorems

Let the fluid be bounded by the free surface F, fixed surfaces S and the surfaces of a finite number of bodies of bounded extent undergoing specified motions of small amplitude about equilibrium positions S_m (see Figure 28). Let the pressure distribution

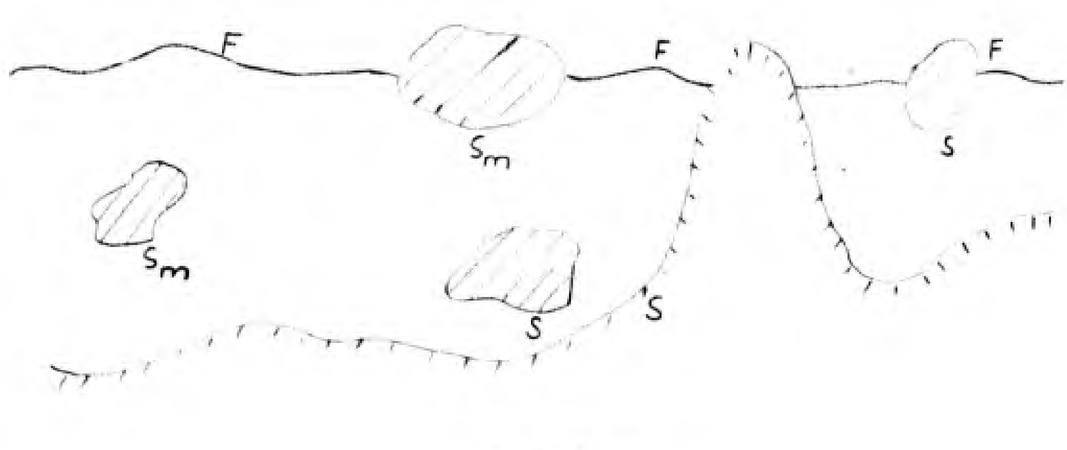

FIGURE 28

on the free surface F, also be a given function $p(x,z,t)$. Furthermore, at time $t = 0$ let the initial displacement and vertical velocity of the free surface be given functions

$$\eta(x,z,0), \quad \eta_t = (x,z,0). \tag{22.1}$$

The boundary conditions to be satisfied by the velocity potential $\phi(x,y,z,t)$ are (see 11.1)

$$\begin{aligned}
&\phi_{tt}(x,0,z,t) + g\,\phi_y(x,0,z,t) = -\frac{1}{\rho} p_t(x,z,t) \quad \text{on } F,\\
&\phi_n = 0 \quad \text{on } S,\\
&\phi_n = V_n(t) \quad \text{on } S_m,\\
&\phi_t(x,0,z,0) = -g\,\eta(x,z,0) - \frac{1}{\rho} p(x,z,0) \quad \text{on } F,\\
&\phi_y(x,0,z,0) = \eta_t(x,z,0) \quad \text{on } F.
\end{aligned} \tag{22.2}$$

Here F means that part of the plane $y = 0$ occupied by fluid when everything is at rest. In addition, it will be assumed that, for each t, there is a bound B and a distance r_0 such that $|\phi|$, $|\phi_t|$, $|\text{grad } \phi|$ and $|\text{grad } \phi_t|$ are each less than B for $x^2+y^2+z^2 > r_0^2$.

Let us now suppose that it is possible to find a source function G of the following nature:

$$G(x,y,z;\xi,\eta,\zeta;t,\tau) = \frac{1}{r} + H(x,y,z;\xi,\eta,\zeta;t,\tau), \tag{22.3}$$

where as usual $r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ and H is harmonic for $y \leq 0$, and in addition G satisfies

$$G(x,y,z;\xi,\eta,\zeta;t,t) = G_t(x,y,z;\xi,\eta,\zeta;t,t) = 0, \tag{22.4}$$

$$G_n(x,y,z;\xi,\eta,\zeta;t,\tau) = 0 \quad \text{for } (x,y,z) \text{ on } S \text{ and for all } t.$$

This function has already been constructed in two cases. If there are no fixed boundaries and the fluid is infinitely deep, the function defined in (13.49) satisfies the conditions after slight modifications: replace (a,b,c) by (ξ, η, ζ), set m(t) = 1, and extend the definition of ϕ (of 13.49) to negative t by $\phi(x,y,z,-t) = \phi(x,y,z,t)$. Then we take

$$G(x,y,z;\xi,\eta,\zeta;t,\tau) = \phi(x,y,z;t-\tau) - G(x,y,z;\xi,\eta,\zeta;\tau,t).$$

Similarly, the function defined in (13.53) allows one to construct G when the fixed boundary consists of a horizontal bottom at y = -h. For the first G. Finkelstein [1957, Appendix] has shown that G is $O(R^{-2})$ and G_R and G_y are $O(R^{-3})$ as $R \to \infty$, where $R^2 = (x-\xi)^2 + (z-\zeta)^2$; for the second G. Finkelstein [1957, §3] has shown that G, G_R and G_y are $o\left(\exp\left(\frac{-\pi}{2h}+\varepsilon\right)R\right)$ as $R \to \infty$ for arbitrary $\varepsilon > 0$.

Now apply Green's theorem to the functions ϕ_t and G and the region of fluid bounded by the surfaces S_m, the fixed boundaries S and a large sphere Ω of radius ρ and center at the origin, where ρ is chosen large enough to include all the surfaces S_m. Only parts of F, S and Ω will serve as bounding surfaces, and we shall call these parts F', S' and Ω', respectively. Then

$$\phi_t(x,y,z,t) = \frac{1}{4\pi}\iint_{F'+S_m+S'+\Omega'} \left[G(\xi,\eta,\zeta;x,y,z;t,\tau)\phi_{t\nu}(\xi,\eta,\zeta,t) - \phi_t G_\nu\right] d\sigma \qquad (22.5)$$

where ν is the exterior normal. The right-hand side is actually independent of τ since τ enters only through the function H which is harmonic. The integral over S' vanishes since both ϕ_n and G_n

are zero on S. We shall assume that the behavior of G and ϕ as $R \to \infty$ is such that the integral over Ω' vanishes as $\rho \to \infty$. If the fluid is of bounded extent, the situation considered by Volterra [1934], this presents, of course, no difficulty. In the two cases for which G has been given above, it has been shown by Finkelstein that this is true. For finite depth the proof presents no difficulty once the estimates for G are obtained; for infinite depth the analysis is more troublesome and we refer to his paper or to Stoker [1957, pp. 193-4] for proof. After letting $\rho \to \infty$, one then has

$$\phi_t(x,y,z,t) = \frac{1}{4\pi}\iint_F [G\phi_{t\eta} - G_\eta \phi_t]\,d\sigma + \frac{1}{4\pi}\iint_{S_m} [G\phi_{t\nu} - G_\nu \phi_t]\,d\sigma. \qquad (22.6)$$

In the integral over F we may replace G_η by $-g^{-1}G_{tt}$ because of the boundary condition at F. Now interchange t and τ and integrate with respect to τ between limits 0 and t. This gives, following an integration by parts,

$$\phi(x,y,z,t) - \phi(x,y,z,0) = \frac{1}{4\pi}\iint_F \left\{ \left[G\phi_\eta + \frac{1}{g}\phi_t G_t\right]\Big|_0^t - \int_0^t \left[G_t\phi_\eta + \frac{1}{g}\phi_{tt} G_t\right] d\tau \right\}$$
$$+ \frac{1}{4\pi}\int_0^t d\tau \iint_{S_m} [G\phi_{tn} - G_n \phi_t]\,d\sigma \qquad (22.7)$$
$$= \frac{1}{4\pi}\iint_F \Big\{ G(\xi,0,\zeta,x,y,z;t,t)\phi_y(\xi,0,\zeta,t) + \frac{1}{g}\phi_t(\ldots,t)G_t(\ldots;t,t)$$
$$- G(\ldots;0,t)\phi_y(\ldots,0) - \frac{1}{g}\phi_t(\ldots,0)G_t(\ldots;0,t)$$
$$+ \frac{1}{\rho g}\int_0^t G_t(\ldots;\tau,t)\,p_t(\xi,\zeta,\tau)\,d\tau \Big\}\,d\xi\,d\zeta + I$$

where I stands for the last integral. (G_t always represents the derivative with respect to the seventh variable). Recalling the

properties of G in (22.4), one finds

$$\phi(x,y,z,t) = \phi(x,y,z,0) + \frac{1}{4\pi}\iint_F \Big\{ -G(\xi,0,\zeta;x,y,z;0,t)\eta_t(\xi,\zeta,0) + \tag{22.8}$$

$$+ G_t(\xi,0,\zeta,x,y,z;0,t)\Big[\eta(\xi,\zeta,0) + \frac{1}{\rho g}p(\xi,\zeta,0)\Big] + \frac{1}{\rho g}\int_0^t G_t(\xi,0,\zeta;x,y,z,\tau,t)\cdot$$

$$\cdot p_t(\xi,\zeta,\tau)d\tau\Big\} d\xi\, d\zeta + I$$

$$= \phi(x,y,z,0) + \frac{1}{4\pi}\iint_F \Big\{ -G(\xi,0,\zeta,x,y,z;0,t)\eta_t(\xi,\zeta,0) + G_t(\xi,0,\zeta,x,y,z;0t)\eta(\xi,\zeta,0)$$

$$- \frac{1}{\rho g}\int_0^t G_{tt}(\xi,0,\zeta,x,y,z;\tau,t)\, p(\xi,\zeta,\tau)d\tau\Big\} d\xi\, d\zeta + I,$$

where $\phi(x,y,z,o)$ is determined up to an additive constant as the solution to a Neumann problem, since $\phi_n(x,y,z,o)$ is given on all boundaries and bounded at infinity. In the integral I we note that $\phi_{tn} = V_n'(t)$ is known on S_m, but ϕ_t is not.

If there are no moving bodies in the fluid, then the integral I is not present and ϕ is determined by the initial displacement and velocity of the free surface and the given pressure distribution over it. This is Volterra's result as extended to unbounded fluids by Finkelstein. If surfaces S_m are present, one may still use (22.8) to derive an integral equation in the same way that (16.13) was derived. For as (x,y,z) is made to approach a point (x_o,y_o,z_o) of S_m,

$$\frac{1}{4\pi}\iint_{S_m} G_\nu(\xi,\eta,\zeta,x,y,z;\tau,t)\phi_t(\xi,\eta,\zeta,\tau)d\sigma \to \frac{1}{2}\phi_t(x_o,y_o,z_o,t) +$$

$$+ \frac{1}{4\pi}\iint_{S_m} G_\nu(\xi,\eta,\zeta,x_o,y_o,z_o;\tau,t)\phi_t\, d\sigma$$

[cf. Kellogg, Foundations of potential theory, Springer, Berlin, 1929, p. 167]. Thus, after carrying out the integration with

respect to τ, one has an integral equation for $\phi(x,y,z,t)$ for each value of $t > 0$ which may be used to find the value of ϕ, and hence ϕ_t, on the surface S_m, providing that the integral equation can be solved. One may then use (22.8) to determine $\phi(x,y,z,t)$ for all values of (x,y,z) in the fluid. The integral equation has the same appearance as (22.8) except that the first two termshave coefficients $\frac{1}{2}$ and (x,y,z) is understood to be a point of S_m. This further extension of Volterra's analysis is also due to Finkelstein.

Uniqueness of $\phi(x,y,z,t)$, at least up to an additive constant, may be proved as follows. Let ϕ_1 and ϕ_2 be two solutions satisfying the boundary conditions. Then $\phi = \phi_1 - \phi_2$ satisfies (22.8) with f, F, p and V_n all identically zero, i.e.

$$\phi(x,y,z,t) = \text{const.} - \frac{1}{4\pi}\int_0^t d\tau \iint_{S_m} G_n \phi_t \, d\sigma .$$

If we assume that G_n is $O(R^{-1-\epsilon})$ as $R \to \infty$, then ϕ_t and grad ϕ will have the same behavior and the integrals we shall write below may be shown to exist. As has been mentioned above, G_n vanishes much quicker than is required in the cases when the fluid is infinitely deep and when the fixed surface consists of a horizontal bottom; if the fluid is bounded in extent, no such condition is necessary to make the integrals converge.

Consider then, following Volterra,

$$\Omega = \frac{1}{2}\frac{\partial}{\partial t}\iint_F \frac{1}{g}\phi_t^2 \, d\sigma$$

$$= \iint_F \frac{1}{g}\phi_t \phi_{tt} \, d\sigma = -\iint_F \phi_t \phi_y \, d\sigma$$

$$= -\iint_{F+S+S_m} \phi_t \phi_n \, d\sigma$$

since ϕ_n vanishes on S and S_m. Now apply Green's theorem and denote the volume occupied by fluid by T:

$$\Omega = -\iiint_T \operatorname{grad}\phi_t \cdot \operatorname{grad}\phi \, d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\iiint_T (\operatorname{grad}\phi^2)\, d\tau.$$

Hence

$$\frac{\partial}{\partial t}\left\{\iint_F \frac{1}{g}\phi_t^2 \, d\sigma + \iiint_T (\operatorname{grad}\phi)^2 d\tau\right\} = 0$$

and

$$\iint_F \frac{1}{g}\phi_t^2 \, d\sigma + \iiint_T (\operatorname{grad}\phi)^2 d\tau = \text{const.} \tag{22.9}$$

Since $\phi_n = 0$ on F, S and S_m for $t = 0$, $\phi(x,y,z,o) = C$, a constant; hence grad $\phi = 0$ for $t = 0$. Also $\phi_t(x,y,z,o) = 0$. Hence the constant in (22.9) is zero and ϕ_t and grad ϕ vanish for all t. Thus $\phi(x,y,z,t)$ = const. and the solution of the initial-value problem is determined up to a constant.

22 β. The Cauchy-Poisson problem

In this classical problem of water-wave theory, the pressure over the free surface is constant, say zero, the fluid is infinitely deep or bounded below by a horizontal bottom, no obstructions are present and the initial displacement and velocity of the free surface are given. The two- and three-dimensional cases will be separated in order to illustrate different methods of approach.

Three dimensions. The velocity potential may be obtained directly from (22.8) after setting $p(x,z,t)$ and I equal to zero. However, the explicit expressions for G and G_t are needed. As was

noted in section 22 α these can be written down immediately from (13.49) for infinite depth and (13.53) for depth h. The resulting expressions, after setting $\eta = 0$, are as follows:

infinite depth:

$$G(x,y,z;\xi,0,\zeta,0,t) = 2\int_0^\infty [1-\cos(\sqrt{gk}\,t)]e^{ky} J_0(kR)\,dk,$$

$$G_t(x,y,z;\xi,0,\zeta;0,t) = -2\int_0^\infty \sin(\sqrt{gk}\,t)e^{ky} J_0(kR)\sqrt{gk}\,dk; \tag{22.10}$$

depth h:

$$G(x,y,z;\xi,0,\zeta;0,t) = 2\int_0^\infty [1-\cos(\sqrt{gk\tanh kh}\,t)]\frac{\cosh k(y+h)}{\sinh kh} J_0(kR),$$

$$G_t(x,y,z;\xi,0,\zeta;0,t) = -2\int \sqrt{gk\tanh kh}\,\sin(\sqrt{gk\tanh kh}\,t)\frac{\cosh k(y+h)}{\sinh kh} J_0(kR \tag{22.11}$$

where $R^2 = (x-\xi)^2 + (z-\zeta)^2$.

There still remains to find $\phi(x,y,z,o)$ where

$$\phi_y(x,0,z,0) = \eta_t(x,z,0)$$

and

$$\lim_{y\to-\infty} \phi_y(x,y,z,0) = 0 \quad \text{or} \quad \phi_y(x,-h,z,0) = 0.$$

The solution of these two problems is well known:

infinite depth:

$$\phi(x,y,z,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\eta_t(\xi,\zeta,0)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{1/2}} = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^\infty e^{ky}J_0(kR)\,dk, \tag{22.12}$$

depth h:

$$\phi(x,y,z,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}\frac{\cosh k(y+h)}{\sinh kh}\,J_0(kR)\,dk, \tag{22.13}$$

where R is defined as above.

Substituting the several expressions in (22.8), one obtains the expressions for the velocity potential:

infinite depth:

$$\begin{aligned}\phi(x,y,z,t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}e^{ky}\cos\sigma t\,J_0(kR)\,dk\\ -\frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi,d\zeta\int_0^{\infty}\sigma\, e^{ky}\sin\sigma t\,J_0(kR)\,dk,\end{aligned} \tag{22.14}$$

$$\sigma^2 = gk\,;$$

depth h:

$$\begin{aligned}\phi(x,y,z,t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}\frac{\cosh k(y+h)}{\sinh kh}\cos\sigma t\,J_0(kR)\,dk\\ -\frac{1}{2\pi}\iint_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}\sigma\,\frac{\cosh k(y+h)}{\sinh kh}\sin\sigma t\,J_0(kR)\,dk,\end{aligned} \tag{22.15}$$

$$\sigma^2 = gk\tanh kh.$$

The equations describing the free surface are as follows:

infinite depth:

$$\begin{aligned}\eta(x,z,t) = \frac{1}{2\pi g}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}\sigma\sin\sigma t\,J_0(kR)\,dk\\ +\frac{1}{2\pi g}\iint_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^{\infty}\sigma^2\cos\sigma t\,J_0(kR)\,dk,\end{aligned} \tag{22.16}$$

$$\sigma^2 = gk\,;$$

depth h:

$$\eta(x,z,t) = \frac{1}{2\pi g}\iint_{-\infty}^{\infty}\eta_t(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^\infty \sigma \sin\sigma t \coth kh\, J_0(kR)\,dk$$

$$+\frac{1}{2\pi g}\iint_{-\infty}^{\infty}\eta(\xi,\zeta,0)\,d\xi\,d\zeta\int_0^\infty \sigma^2\cos\sigma t \coth kh\, J_0(kR)\,dk, \qquad (22.17)$$

$$\sigma^2 = gk \tanh kh.$$

It has been shown by Kochin [1935] that the integrals with respect to k in (22.16) can be evaluated. Consider the integral

$$K = \int_0^\infty \sigma^{-1}\sin\sigma t\, J_0(kR)\,dk, \quad \sigma^2 = gk. \qquad (22.18)$$

Then the first integral with respect to k in (22.16) is $-K_{tt}$ and the second one is $-K_{ttt}$. To evaluate K make first the following change of variables:

$$\varkappa^2 = kR, \quad \omega^2 = gt^2/4R. \qquad (22.19)$$

Then

$$K = \frac{2}{\sqrt{gR}}\int_0^\infty \sin 2\omega\varkappa\, J_0(\varkappa^2)\,d\varkappa$$

$$= \frac{2}{\sqrt{gR}}\int_0^\infty d\varkappa\int_0^1 \frac{\sin 2\omega\varkappa\cos v\varkappa^2}{\sqrt{1-v^2}}\,dv$$

$$= \frac{2}{\sqrt{gR}}\int_0^\infty d\varkappa\int_0^1 \left[\sin(2\omega\varkappa + v\varkappa^2) + \sin(2\omega\varkappa - v\varkappa^2)\right]\frac{dv}{\sqrt{1-v^2}}.$$

In the first integral let $u = v\varkappa + \omega$, in the second let $u = v\varkappa - \omega$.

Then

$$K = \frac{2}{\sqrt{gR}} \int_{+\omega}^{\infty} du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}} - \frac{2}{\sqrt{gR}} \int_{-\omega}^{\infty} du \int_0^1 \sin\left(\frac{u^2-\omega^2}{v}\right) \frac{dv}{v\sqrt{1-v^2}}$$

$$= + \frac{2}{\sqrt{gR}} \int_{-\omega}^{\omega} du \int_1^{\infty} \frac{\sin(\omega^2-u^2)v'}{\sqrt{v'^2-1}} \qquad (v' = 1/v)$$

$$= \frac{4}{\sqrt{gR}} \frac{\pi}{2} \int_0^{\omega} J_0(\omega^2-u^2)\, du,$$

and, after setting $u = \sqrt{2}\ \omega \sin\frac{1}{2}\theta$,

$$K = \frac{\sqrt{2}\,\pi}{\sqrt{gR}} \omega \int_0^{\frac{1}{2}\pi} J_0\left(2\frac{\omega^2}{2}\cos\theta\right) \cos\tfrac{1}{2}\theta\, d\theta.$$

Finally, from an identity in Watson's Bessel functions [§ 5.43, eq. (1)] one finds

$$K(\omega) = \frac{\pi^2}{\sqrt{2gR}} \omega J_{\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right). \tag{22.20}$$

In order to use the result in (22.16) one needs the first three derivatives with respect to t. Since

$$\frac{\partial}{\partial t} = \frac{1}{2}\sqrt{\frac{g}{R}}\, \frac{\partial}{\partial \omega},$$

the derivatives can be computed by taking derivatives with respect to ω and multiplying by an appropriate factor. After some rather tedious computation one finds

$$\frac{\partial}{\partial t} K(\omega) = -\frac{1}{2\sqrt{2}} \pi^2 \frac{\omega^2}{R} \left[J_{\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right) J_{\frac{3}{4}}\left(\tfrac{1}{2}\omega^2\right) - J_{-\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right) J_{-\frac{3}{4}}\left(\tfrac{1}{2}\omega^2\right)\right],$$

$$\frac{\partial^2}{\partial t^2} K(\omega) = -\frac{1}{2} \pi^2 \omega^3 \sqrt{\frac{g}{2R^3}} \left[J_{\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right) J_{-\frac{1}{4}}\left(\tfrac{1}{2}\omega^2\right) + J_{\frac{3}{4}}\left(\tfrac{1}{2}\omega^2\right) J_{-\frac{3}{4}}\left(\tfrac{1}{2}\omega^2\right)\right], \tag{22.21}$$

$$\frac{\partial^3}{\partial t^3} K(\omega) = -\frac{\pi^2 g}{2R^2}\,\omega^2\Big[J_{\frac{1}{4}}(\tfrac{1}{2}\omega^2)\,J_{-\frac{1}{4}}(\tfrac{1}{2}\omega^2) - \omega^2\Big\{J_{\frac{1}{4}}(\tfrac{1}{2}\omega^2)\,J_{\frac{3}{4}}(\tfrac{1}{2}\omega^2) - J_{-\frac{1}{4}}(\tfrac{1}{2}\omega^2)\,J_{\frac{3}{4}}(\tfrac{1}{2}\omega^2)\Big\}\Big].$$

These are Kochin's formulas, but derived somewhat differently from his original paper; still another derivation may be found in Kochin, Kibel and Roze [1948, ch. 8, § 21]. Similar formulas for (22.17) do not seem to have been discovered.

It should be noted that in the find form of (22.16) the dependence upon t is through the dimensionless variable $\omega^2 = gt^2/4R$. Hence, if one examines the contribution to the surface profile from a given locality, say the neighborhood of (ξ, ζ), then a given phase of this contribution, say a maximum, will be described by $gt^2/4R$ = const.; i.e., the phase is moving away from (ξ, ζ) with constant acceleration proportional to g. The amplitude of the contribution is modulated by either $R^{-3/2}$ or R^{-2} according as one is considering the first or second summand in (22.16). Kochin's 1935 paper is of some methodological interest inasmuch as he started his analysis with dimensional considerations. This method will be introduced for the two-dimensional case.

One may obtain without great difficulty series expansions for the k-integrals in (22.14) and (22.16), as was first done by Cauchy and Poisson. We refer to Lamb's Hydrodynamics [1932, § 255] for the derivation and exact expressions. They can also be derived from the the known expansions for $J_{\frac{1}{4}}$, etc., as can asymptotic expressions for large ω. One may also carry out an analysis of the changing shape of the surface profile following the methods of section 15.

It is evident that one can solve explicitly other similar

initial-value problems for which the Green's function can be given. For example, the method of images allows one to give an explicit solution for various cases when vertical walls are present as boundaries. Such cases have been considered by Risser [1925]. The Cauchy-Poisson problem in the presence of a vertical half-plane, $z = 0$, $x > 0$, has been treated by Boiko [1938], but by more complex methods.

Two dimensions. Rather than repeat the methods used for three-dimensional motion, we shall introduce a method making use of the complex potential and thus special to two-dimensional motion. It is analogous to the method used in deriving (13.28).

Let $f(z,t) = \phi(x,y,t) + i\psi(x,y,t)$ be the complex velocity potential. The initial conditions will be taken in the form

$$-\frac{1}{g}\,\mathrm{Re}\,f_t(x-i0,0+) = \eta(x,0), \quad -\mathrm{Im}\,f'(x-i0,0+) = \eta_t(x,0). \tag{22.22}$$

Let us consider infinite depth first. For $t > 0$ we assume that $f(z,t)$ is regular and $|f'| < M(t)$, $|f_{tt}| < M(t)$ for $y < 0$ and that both f' and f_{tt} approach zero as $y \to -\infty$. Consider now the function

$$G(z,t) = f_{tt}(z,t) + ig\,f'(z,t). \tag{22.23}$$

From the assumptions about f it follows that, for $t > 0$, $G(z,t)$ is regular for $y < 0$, that $|G| < B(t)$ for $y < 0$ and that $G \to 0$ as $y \to -\infty$. Moreover, it follows from the condition at the free surface, (11.5), that $\mathrm{Re}\,G(x-i0,t) = 0$. Hence, the definition of G may be extended into the upper half-plane by defining

$$G(x+iy) = -\overline{G(x-iy)}. \tag{22.24}$$

But then since G is regular and bounded in the whole finite z-plane, it follows from Liouville's theorem that G = const., the constant must equal zero from the assumed behavior as $y \to -\infty$. Hence the fundamental differential equation for the Cauchy-Poisson problem in two dimensions is

$$f_{tt}(z,t) + igf'(z,t) = 0 \quad , \quad t > 0 \ , \tag{22.25}$$

an observation usually credited to Levi-Civita [cf. Tonolo, 1913].

Let us now find the analogous equation for finite depth. The function f will be assumed regular in the strip $0 > y > -h$. The boundary condition on the bottom is

$$I_m f'(x-ih,t) = 0. \tag{22.26}$$

Hence f may be extended analytically into the strip $-2h < y < -h$ by defining

$$f(x-i(y+2h)) = \overline{f(x+iy)}, \quad 0 > y \geqslant -h. \tag{22.27}$$

We may also, as before, extend the function $f_{tt}+igf'$ into the strip $h \geqslant y \geqslant 0$. The condition $\mathrm{Re}\left\{f_{tt}+igf'\right\} = 0$ for $y = 0$ implies $\mathrm{Re}\left\{f_{tt}-igf'\right\} = 0$ for $y = -2h$. Hence the function $f_{tt}-igf'$ can be extended by reflection into the strip $-3h \leqslant y \leqslant -2h$. Now consider the function

$$H(z,t) = [f_{tt}(z+ih,t) + f_{tt}(z-ih,t)] + ig[f'(z+ih,t) - f'(z-ih,t)]$$

$$= \{f_{tt}(z+ih,t) + igf'(z+ih,t)\} + \{f_{tt}(z-ih,t) - igf'(z-ih,t)\}. \tag{22.28}$$

As a result of the various extended regions of definition, one may verify easily that H is defined for all z in the strip $-2h < y < 0$ and is regular there. Moreover, it follows that

$$H(x-ih,t) = 0 \; ; \tag{22.29}$$

for from (22.27) it follows that the two pairs of summands in the first form of (22.28) are real for $z = x-ih$, whereas from the boundary conditions at $y = 0$ and $y = -2h$ it follows that the terms in curly brackets in the second form of (22.28) have zero real parts. Since $H(z,d)$ is regular in the strip $0 > y > -2h$ and vanishes on $y = -h$, it must vanish identically in the strip. Hence we have the following differential-difference equation of Cisotti [1918]:

$$f_{tt}(z+ih,t) + f_{tt}(z-ih,t) + ig[f'(z+ih,t) - f'(z-ih,t)] = 0. \tag{22.30}$$

Let us now turn to the solution of (22.25) with initial conditions (22.22). We shall follow closely an exposition of Sedov's [1948, 1957]. However, the idea of the derivation is Kochin's [1935] and, in fact, really goes back to Tonolo [1913]. The use of dimensional analysis can be extended to the three-dimensional problem; this was also done by Kochin.

We first remark that the initial-value problem can be solved

by solving it for two special cases of (22.22), namely, first with $\eta(x,0) = 0$ and then with $\eta_t(x,0) = 0$. The sum of these two solutions will satisfy (22.22). Next we note that $\eta(x,0)$ has the dimension "length" and $\eta_t(x,0)$ the dimension "velocity", and that the solution f in each of the two initial-value problems will be proportional to some typical parameters associated with $\eta(x,0)$ or $\eta_t(x,0)$, respectively. Let us suppose that a is such a parameter with dimension $L^p T^q$ and that f is proportional to a. Since f has dimension L^2T^{-1} and g has dimension LT^{-2}, the Pi theorem of dimensional analysis then states that f can be expressed as follows:

$$f(z,t) = a z^{\alpha} g^{\beta} \chi\left(\frac{i g t^2}{4z}\right), \tag{22.31}$$

where

$$\alpha = \frac{5}{2} - p - \frac{1}{2} q \;, \quad \beta = \frac{1}{2}(q+1). \tag{22.32}$$

(The factor i/4 in the argument of χ is chosen for later convenience.) Now substitute (22.31) into (22.25). One finds after some computation that

$$f_{tt} + i g f' = i a z^{\alpha-1} g^{\beta+1} [\zeta \chi''(\zeta) + (\tfrac{1}{2} - \zeta)\chi'(\zeta) + \alpha \zeta] = 0, \tag{22.33}$$

where $\zeta = igt^2/4z$. The differential equation obtained by setting the expression in square brackets equal to zero determines χ in terms of confluent hypergeometric functions:

$$\chi(\zeta) = A\, {}_1F_1(-\alpha, \tfrac{1}{2}; \zeta) + B\, \zeta^{\frac{1}{2}}\, {}_1F_1(\tfrac{1}{2} - \alpha, \tfrac{3}{2}; \zeta). \tag{22.34}$$

From this it follows that

$$f(\zeta,t;\alpha) = a\zeta^{\alpha} g^{\beta}\left[A\,{}_1F_1\left(-\alpha,\tfrac{1}{2};\frac{igt^2}{4\zeta}\right) + B\left(\frac{igt^2}{4\zeta}\right)^{\frac{1}{2}}{}_1F_1\left(\tfrac{1}{2}-\alpha,\tfrac{3}{2};\frac{igt^2}{4\zeta}\right)\right] \tag{22.35}$$

$$= A f_1(\zeta,t;\alpha) + B f_2(\zeta,t;\alpha).$$

Remembering that

$${}_1F_1(\gamma,\delta;0) = 1, \quad {}_1F_1'(\gamma,\delta;0) = \gamma/\delta,$$

one may easily derive the following:

$$f(\zeta,0) = A f_1(\zeta,0) = A a g^{\beta}\zeta^{\alpha},$$

$$f'(\zeta,0) = A f_1'(\zeta,0) = A a \alpha g^{\beta}\zeta^{\alpha-1}, \tag{22.36}$$

$$f_t(\zeta,0) = B f_{2t}(\zeta,0) = \tfrac{1}{2} B a i^{\frac{1}{2}} g^{\beta+\frac{1}{2}} \zeta^{\alpha-\frac{1}{2}}.$$

The solution (22.35) may be further generalized by replacing t by $t-t_o$ and z by $z-x_o$ (i.e., by a different choice of the dimensionless variable ζ). One may then further superimpose these solutions. For the purpose at hand it will be sufficient to retain $t_o = 0$. Then we may form the solution

$$f(\zeta,t) = \int_{-\infty}^{\infty} A(x_o) f_1(\zeta - x_o, t;\alpha_1)\,dx_o + \int_{-\infty}^{\infty} B(x_o) f_2(\zeta - x_o, t;\alpha_2)\,dx_o. \tag{22.37}$$

One finds from (22.30) that

$$f(\zeta,0) = a_1 g^{\beta_1}\int_{-\infty}^{\infty} A(x_o)(\zeta - x_o)^{\alpha_1}\,dx_o,$$

$$f'(\zeta,0) = a_1\alpha_1 g^{\beta_1}\int_{-\infty}^{\infty} A(x_o)(\zeta - x_o)^{\alpha_1 - 1}\,dx_o, \tag{22.38}$$

$$f_t(\zeta,0) = \tfrac{1}{i}\, a_2\, i^{\frac{1}{i}}\, g^{\beta_2+\frac{1}{i}} \int_{-\infty}^{\infty} B(x_0)(\zeta - x_0)^{\alpha_2-\frac{1}{i}}\, dx_0 .$$

Let us now make some special choices of a, and hence of α and β. As a parameter describing the initial profile of the surface take

$$a_2 = \int_{-\infty}^{\infty} \eta(x,0)\, dx \; ; \tag{22.39}$$

as a parameter describing the initial velocity distribution take

$$a_1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \eta_t(\xi,0)\, d\xi . \tag{22.40}$$

Then a_1 has the dimension L^3T^{-1}, corresponding to $\alpha_1 = -1$, $\beta_1 = 0$, and a_2 the dimension L^2, corresponding to $\alpha_2 = -\frac{1}{2}$, $\beta_2 = \frac{1}{2}$. With these choices of α_1 and α_2 in (22.37) we take

$$\begin{aligned} A(x_0) &= \frac{-1}{a_1 \pi} \int_{-\infty}^{x_0} \eta_t(\xi,0)\, d\xi , \\ B(x_0) &= \frac{-2}{a_2\, \pi\, i^{3/2}}\, \eta(x_0,0). \end{aligned} \tag{22.41}$$

Then the last two equations of (22.38) become (after an integration by parts in the first one)

$$f'(\zeta,0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta_t(x_0,0)}{x_0 - \zeta}\, dx_0 \, ,$$

$$f_t(\zeta,0) = \frac{g}{\pi i} \int_{-\infty}^{\infty} \frac{\eta(x_0,0)}{x_0 - \zeta}\, dx_0 \, .$$

From the Plemelj-Sokhotskii theorem we have

$$f'(x - i0, 0) = -i\, F(x) + \frac{1}{\pi}\, PV\!\!\int_{-\infty}^{\infty} \frac{\eta_t(x_0,0)}{x_0 - x}\, dx_0 \, , \tag{22.42}$$

$$f_t(x-i0,0) = -g\,\ell(x) + \frac{g}{\pi i}\ \mathrm{PV}\int_{-\infty}^{\infty} \frac{\eta(x_0,0)}{x_0 - x}\, dx_0 .$$

Thus the initial conditions (22.22) are satisfied.

There remains to point out that for the special choices of $\alpha_1 = -1$ and $\alpha_2 = -\frac{1}{2}$ the corresponding confluent hypergeometric functions in (22.35) may be expressed in terms of Fresnel integrals or integrals of these. In fact, if we write (22.37) in the form

$$f(z,t) = \int_{-\infty}^{\infty} \Omega_1(z-x_0,t)\int_{-\infty}^{x_0} \eta_t(\xi,0)\,d\xi\, x_0 + \int_{-\infty}^{\infty} \Omega_2(z-x_0,t)\,\eta(x_0,0)\,dx_0, \tag{22.43}$$

then

$$\Omega_1(z,t) = -\frac{2i}{z}\, e^{-i\frac{\pi}{2}\omega^2}\,\omega - \frac{2i}{z}\int_0^{\omega} du \int_0^{u} e^{-i\frac{\pi}{2}(v^2-\omega^2)}\, dv,$$

$$\Omega_2(z,t) = 2i\sqrt{\frac{2g}{\pi z}}\int_0^{\omega} e^{-i\frac{\pi}{2}(u^2-\omega^2)}\, du, \tag{22.44}$$

where

$$\omega^2 = \frac{g t^2}{2\pi z}.$$

One should also consult the discussion in Lamb's Hydrodynamics [1932, § 238, 239], where graphs are given which display the behavior of the surface profile corresponding to an initial elevation concentrated in the neighborhood of one point, i.e., essentially $-g^{-1}\Omega_{2t}(x-i0,t)$, and to a concentrated impulse, i.e., essentially $-g^{-1}\Omega_{1t}(x-i0,t)$. However, general aspects of the development of the surface profile have already been discussed in section 15α.

It should be noted that the velocity potential (22.37) represents a much wider class of time-dependent gravity-wave motions

than does (22.43). The initial-value problems corresponding to other valves of α have been determined by Sedov [1948] but the discussion will not be repeated here.

A class of solutions of (22.30) analogous to that found by Sedov for (22.25) does not seem to have been given in the published literature. Cisotti [1920] expands f(z,t) in a power series in t, thus replacing (22.30) by a recursive set of difference equations. We refer to the original paper for his discussion of this set of equations.

In (15.22) we have already given the velocity potential and surface profile corresponding to a given initial profile; the derivation was based upon a Fourier analysis of the initial profile and the result was valid for either finite or infinite depth. The same procedure may be used for an initial velocity distribution. The combined result for the complex velocity potential and surface profile is given below in such a way as to include the possible presence of surface tension:

infinite depth:

$$f(z,t) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \, \frac{1}{k}\left[-\sigma(k)\eta(\xi,0)\sin\sigma t + \eta_t(\xi,0)\cos\sigma t\right] e^{ik(z-\xi)}$$

$$\eta(x,t) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \left[\eta(\xi,0)\cos\sigma t + \frac{\sigma}{gk}\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi), \tag{22.45}$$

where $\sigma^2 = gk + Tk^3/\rho$;

depth h:

$$f(z,t) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty d\xi \, \frac{1}{k\sinh kh}\left[-\sigma(k)\eta(\xi,0)\sin\sigma t + \right.$$
$$\left. + \eta_t(\xi,0)\cos\sigma t\right]\cos k(z-\xi+ih), \tag{22.46}$$

$$\eta(x,t) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{\infty} d\xi \left[\eta(\xi,0)\cos\sigma t + \frac{\sigma(k)}{gk\tanh kh}\,\eta_t(\xi,0)\sin\sigma t\right]\cos k(x-\xi),$$

where $\sigma^2 = (gk + Tk^3/\rho)\tanh kh.$

If T = 0, the coefficients of $\eta_t(\xi,0)$ in the formulas for $\eta(x,t)$ reduce to σ^{-1}. When T = 0, (22.45) is, of course, another form of (22.43).

It has already been indicated in section 15 that the Cauchy-Poisson problem can also be solved for two-fluid problems. Sretenskii [1955] has investigated a further generalization in which the two fluids are each flowing with constant velocities for $t < 0$ and then when $t = 0$ a disturbance is suddenly created at the interface.

22γ. Some other time-dependent problems

It is possible to solve a number of initial-value problems either by using equation (22.8) or by using the time-dependent Green functions (13.49) or (13.53) directly. The special situations treated, below fall roughly into the following four categories: wave motions resulting from a pressure distribution suddenly brought into existence at time t = 0; waves resulting from a body set into motion at time t = 0; waves resulting from an underwater explosion or a sudden movement of the bottom (tsunamis); and waves resulting from an initially displaced freely floating body.

Time-dependent pressure distributions. Suppose that the fluid is undisturbed for $t < 0$ and that starting with $t = 0$ the pressure over the free surface is a given function $p(x, z, t)$. The consequent motion of the fluid may be easily obtained, for this is just the problem formulated in (22.2) if we put $\eta(x, z, 0) = \eta_t(x, z, 0) = 0$. Formula (22.8) then gives the velocity potential in the form

$$\phi(x,y,z,t) = \frac{-1}{4\pi\rho g}\iint_F d\xi\, d\zeta \int_0^t G_{tt}(\xi,0,\zeta,\, x,y,z;\tau,t)\, p(\xi,\zeta,\tau)\, d\tau + I \qquad (22.47)$$

In the two situations for which explicit Green's functions have been given, equations (22.10) and (22.11), we may give explicit solutions for ϕ:

infinite depth:

$$\phi(x,y,z,t) = \frac{-1}{2\pi\rho}\iint_{-\infty}^{\infty} d\xi\, d\zeta \int_0^t p(\xi,\zeta,\tau)\, d\tau \int_0^{\infty} \cos\left(\sqrt{gk}(\tau - t)\right) e^{ky} J_0(kR)\, k\, dk; \qquad (22.48)$$

depth h:

$$\phi(x,y,z,t) = \frac{-1}{2\pi\rho}\iint_{-\infty}^{\infty} d\xi\, d\zeta \int_0^t p(\xi,\zeta,\tau)\,d\tau \int_0^{\infty} \cos\left(\sqrt{gk\tanh kh}\,(\tau - t)\right)\cdot$$

$$\cdot\frac{\cosh k(y+h)}{\cosh kh} J_0(kR)\,k\,dk, \tag{22.49}$$

where, as usual, $R^2 = (x-\xi)^2 + (z-\zeta)^2$.

The velocity potential for a moving pressure distribution is obtained from these expressions simply by letting

$$p(\xi,\zeta,\tau) = p_0(\xi - c\tau, \zeta, \tau).$$

If $p_0(\xi - c\tau, \zeta, \tau) = p_0(\xi - c\tau, \zeta)\cos\sigma\zeta$ the resulting ϕ is the velocity potential for a steadily moving pressure distribution of oscillating strength. Lunde [1951b] has investigated the special case when $p(\xi,\zeta,\tau) = p\left(\sqrt{(\xi - c\tau)^2 + \zeta^2}\right)$ and has shown that as $t \to \infty$ the expressions (22.48) and (22.49), after a change to moving coordinates, approach asymptotically to the expressing (21.26) or (21.31) properly modified for circular symmetry (the assumed symmetry is not essential). The computation is interesting but will not be carried out here. This procedure for obtaining (21.26) or (21.31) yields the velocity potential without necessitating the extra boundary condition requiring the motion to vanish as $x \to +\infty$.

As was mentioned in connection with the solution of the Cauchy-Poisson problem, the Green's function for some other simple configurations can be found by the method of reflection.

The complex velocity potentials for two-dimensional motion which correspond to (22.48) and (22.49) are as follows:

infinite depth:

$$f(\mathfrak{z},t) = \frac{-1}{\pi\rho}\int_{-\infty}^{\infty} d\xi \int_0^t p(\xi,\tau)\,d\tau \int_0^{\infty} \cos(\sqrt{gk}\,(\tau-t))\, e^{-ik(\mathfrak{z}-\xi)}\,dk\,; \tag{22.50}$$

depth h:

$$f(\mathfrak{z},t) = \frac{-1}{\pi\rho}\int_{-\infty}^{\infty} d\xi \int_0^t p(\xi,\tau)\,d\tau \int_0^{\infty} \cos(\sqrt{gk\tanh kh}\,(\tau-t))\,\frac{\cos k(\mathfrak{z}-\xi+ih)}{\cosh kh}\,dk\,. \tag{22.51}$$

Certain special cases have been investigated in more detail. Stoker [1953] has treated the motion resulting when a pressure distribution constant in time for $t > 0$ is suddenly applied to a uniformly moving stream of depth h. The velocity potential may be obtained from (22.51) by taking $p(\xi,\tau) = p_0(\xi - c\tau)$ and transferring to moving coordinates. His aim, as was that of Lunde in the computations described above, was to show that the potential (21.40) can be derived without a special assumption about its behavior as $x \to +\infty$. The same can be carried through with (22.50) to derive (21.38). If one assumes $p(\xi,\tau) = p_1(\xi)\cos\sigma\tau + p_2(\xi)\sin\sigma\tau$, then one may also derive (21.21) or (21.23) from (22.49) or (22.50), respectively, as asymptotic expressions for large t without having to impose a radiation condition. Voit [1957b] has investigated the surface profile for large t when $p(\xi,\tau) = p(\tau)$ for $\xi < c\tau < cT$, $p(\xi,\tau) = 0$ for $\xi \geq c\tau$ or for $\tau > T$.

Waves resulting when a body is set into motion. Many of the problems solved in sections 17-20 by means of source distributions can be formulated as initial-value problems and solved by the same

procedure if one uses the appropriate time-dependent Green's function. We shall consider briefly several examples, omitting details.

In (19.28) the velocity potential was given for the motion resulting from an oscillator in a wall, described by (19.26). It was assumed there that a steady situation had been reached in which the motion was purely harmonic in the time. Suppose instead that the motion of the oscillator described by (19.26) is to start at $t = 0$ and that for $t < 0$ the oscillator and fluid are at rest. It is easy to verify that the time-dependent velocity $\phi(x,y,z,t)$ potential is still given by (19.28) if for the Green's function G one uses (13.50) with $m = 1$. The last term in (13.50) will give the transient aspects of the motion. For two-dimensional motion the time-dependent wave-maker has been considered by Kennard [1949], who also gives an estimate of time necessary for the transient terms to die out.

In (20.65) the velocity potential has been given for a "thin" ship moving with constant velocity c: it is assumed there that a steady state has been reached. Let us now suppose the same ship to move with velocity $c(t)$, $t > 0$, but that it and the fluid have been at rest for $t < 0$. As in (20.64) we take a coordinate system moving with the ship. Then from (20.26) it follows that the velocity potential $\phi(x,y,z,t)$ must satisfy the boundary condition

$$\phi_z(x,y,\pm 0,t) = \mp c(t) F_x(x,y).$$

A Green's function enabling us to construct ϕ can be easily obtained from either (13.49) or (13.53). However, let us take $c(t) = C$, a constant, for $t > 0$, i.e. we suppose the ship to attain instantaneously its final velocity. The Green's function for this

situation has already been written out explicitly in (13.51). Setting there $u_0 = c$, $a_0 = \xi$, $b_0 = \eta$, $c_0 = \zeta$ and calling the resulting function $G(x,y,z,\xi,\eta,\zeta,t)$, the velocity, potential for the problem at hand is

$$\phi(x,y,z,t) = \frac{c}{2\pi}\iint_{S_0} G(x,y,z,\xi,\eta,0,t)\,F_x(\xi,\eta)\,d\xi\,d\eta \qquad (22.52)$$

Having found ϕ, one may then compute the force upon the ship and obtain formulas analogous to (20.67) or (20.69). The computations for infinite depth was originally made by Sretenskii [1937]; Lunde [1951a] gives an exposition of this result and extends it to include thin ships moving in an infinite expanse of fluid of depth h and down the center of a canal of width b and depth h. In these computations c is allowed to be an arbitrary function of t. We refer to Lunde's paper for the results.

Havelock [1948, 1949] has considered the accelerated motion of a submerged horizontal circular cylinder in fluid of infinite depth. The complex velocity potential is expanded in a Laurent series about the center, starting with a dipole. In order to satisfy the other boundary conditions, one makes use of (13.54) to obtain singularities of the proper sort. The boundary condition on the circle then yields an infinite set of equations for determining the coefficients in the Laurent series. After finding as many terms as seems necessary for a suitable approximation, one may compute the force on the cylinder. Havelock has carried this out for the first two singularities (a slight inconsistency in the approximation is corrected in Maruo [1957]) and has made numerical computations for an impulsive

start and for a constant acceleration. Consider an impulsive start with instantaneous acceleration to constant speed c, and let the cylinder have radius $\underline{a}$ and its center be submerged to depth h. Then the two leading terms in the resistance are given by R_0, the steady-state resistance given in equation (20.52), and by the transient term

$$R_1 = \frac{1}{2}\pi g \rho a^4 \nu^2 \left(\frac{\pi}{\nu c t}\right)^{1/2} e^{-1/2 \nu h} \sin\left(\frac{1}{4}\nu c t - \frac{\pi}{4}\right), \quad \nu = \frac{g}{c^2}. \tag{22.53}$$

Figure 28, taken from Maruo [1957], shows R_1/R_0 plotted against ct/h for $c/\sqrt{gh} = 1$.

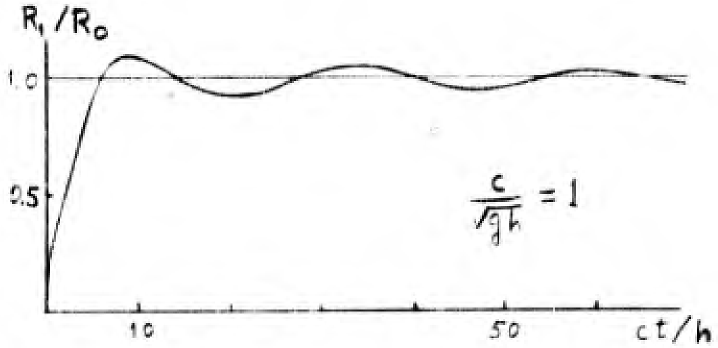

FIGURE 28

An exposition of the theory of accelerated motion of submerged bodies is given by Maruo [1957, ch. 3]. Both two- and three-dimensional problems in fluid of infinite or finite depth are considered.

We note that the use of Kochin's H-function may be extended with no difficulty to time-dependent motion; this has been done by Maruo and earlier by Haskind [1946b].

An investigation of Palm [1953] also fits into the category of problems under consideration. In considering flow over an uneven bottom in section 20 α, it was necessary to impose an upstream boundary condition in order to obtain uniqueness if the velocity is subcritical. In order to avoid this extra condition he formulated an initial-value problem in which the fluid is at rest and the bottom suddenly starts to move. The asymptotic expression for large t in a coordinate system moving with the bottom agrees with the results in 20 α.

<u>Tsunamis and submarine explosions</u>. A tsunami is an ocean wave originating from a sudden upheaval or recession of the ocean floor. If one assumes an ocean of uniform depth h and if the disturbance occurs in a region S of the bottom, one may approximate this situation by the boundary-value problem in which

$$\phi_y(x,-h,z,t) = \begin{cases} V(x,z,t), & 0<t<T,\ (x,z) \text{ in } S, \\ 0, & \text{otherwise.} \end{cases} \tag{22.54}$$

If the time-interval of the disturbance is short (i.e., if gT^2/h is small), the solution for ϕ is given approximately by distributing over S sources of a form easily derived from (13.53). In fact, in (13.53) let $a=\xi$, $b=-h$, $c=\zeta$, and let $2m(t)=2m(\xi,\zeta,t)=-\phi_y(\xi,-h,\zeta,t)/2\pi$; denote the resulting function by $\phi_s(x,y,z,\xi,-h,\zeta,t)$. Then

$$\phi(x,y,z,t) = \iint_S \phi_s(x,y,z,\xi,-h,\zeta,t)\,d\xi\,d\zeta \tag{22.55}$$

is the approximate solution. If one assumes $V(x,\zeta,t) = V(x,\zeta)$ for $0 < t < T$, then ϕ_s takes the following simple form for $t > T$:

$$\phi_s(x,y,\zeta,\xi,-h,\zeta,t) = -\frac{1}{2\pi} V(\xi,\zeta) \int_0^\infty \frac{\cosh k(y+h) J_0(kR)}{\sinh kh \cosh kh} [\cos\sigma(t-T) - \cos\sigma t]\, dk, \tag{22.56}$$

where $\sigma^2 = gk \tanh kh$. If the deformation is assumed to take place so quickly that one may let $T \to 0$ while keeping $VT = L(\xi,\zeta)$ constant (i.e., keeping the same total deformation), (22.56) becomes

$$\phi_s(x,y,\zeta;\xi,-h,\zeta,t) = \frac{-1}{2\pi} L(\xi,\zeta) \int_0^\infty \frac{\cosh k(y+h) J_0(kR)}{\sinh kh \cosh kh} \sigma(k) \sin\sigma(k)t\, dk, \tag{22.57}$$

and the solution (22.55) is no longer approximate for the formulated problem.

A further approximation may be obtained by assuming the area of disturbance to be so localized that one may assume the whole disturbance to originate at one point, say $(0,-h,0)$. Then (22.55) becomes simply (22.57) with L replaced by $Q = \iint L\, d\xi\, d\zeta$ and $R^2 = x^2 + \zeta^2$. Although this may be a reasonable approximation to the explosion of a mine on the ocean floor, it is not in general suitable for a tsunami since the diameter of the region of disturbance in the latter may be many times the depth of fluid.

A comparison of (22.55), with (22.57) for ϕ_s, with (22.15) shows that one may expect the same qualitative behavior for tsunamis as for waves resulting from an initial deformation of the free surface. In fact, if one makes the substitution (22.19) in the expressions for the surface profiles, they become the following, respectively, for the initially displaced surface and the tsunami:

$$\eta(x,z,t) = \frac{1}{\pi}\iint_{-\infty}^{\infty} \eta(\xi,\zeta,0)\frac{1}{R^2}\,d\xi\,d\zeta\int_0^{\infty} \varkappa^3 J_0(\varkappa^2)\cos\left(2\omega\varkappa\sqrt{\tanh \varkappa^2\frac{h}{R}}\right)d\varkappa,$$

$$\eta(x,z,t) = \frac{1}{\pi}\iint_{-\infty}^{\infty} L(\xi,\zeta)\frac{1}{R^2}\,d\xi\,d\zeta\int_0^{\infty} \varkappa^3\frac{J_0(\varkappa^2)}{\cosh\left(\varkappa^2\frac{h}{R}\right)}\cos\left(2\omega\varkappa\sqrt{\tanh \varkappa^2\frac{h}{R}}\right)d\varkappa \tag{22.58}$$

One may study the development of η along the lines worked out in section 15 for two dimensions.

Many of the investigations of tsunamis have been devoted to an examination of the profile for a given type of initial bottom disturbance. The classical papers on tsunamis are by Sano and Hasegawa [1915] and Syono[1936]. They have recently been investigated by Takahashi [1942, 1945, 1947], Ichiye [1950], Gazaryan[1955] and others. Since the shape of the bottom and the configuration of the shore are of obvious importance in a geophysical application of the theory, much recent attention has been given to this aspect of the propagation of tsunamis. Grigorash [1957a] has given a brief survey of the literature together with a substantial bibliography.

The waves resulting from an exploding submerged mine may be represented approximately by using the velocity potential for a source whose strength $m(t)$ has the form of a square pulse of duration T. One may then determine ϕ from either (13.49) or (13.53). If one assumes T very small and forms the limit as $T \to 0$ while keeping $mT = Q$ constant, one finds easily the following expressions for ϕ :

infinite depth:

$$\phi = 2Q\int_0^{\infty} e^{k(y+b)} J_0(kR)\,\sigma(k)\sin\sigma t\,dk, \quad \sigma^2 = gk; \tag{22.59}$$

depth h:

$$\phi = 2Q\int_0^\infty \frac{\cosh k(h+b)\cosh k(y+h)}{\sinh kh \cosh kh}\,\sigma(k)\sin\sigma t\,dk, \quad \sigma^2 = gk\tanh kh. \tag{22.60}$$

Again one may examine the development of the surface profile by the methods developed in section 15.

Investigations of waves generated by a sudden pulse of the above or similar sort have been made by Lamb [1913, 1922] and Terazawa [1915]; both took the fluid to be infinitely deep. Sretenskii [1950, 1949] has made a similar study when the source (two-dimensional) is situated on the bottom of a rectangular basin and within a fluid layer covering a solid sphere. Sezawa [1929a,b] has included the effect of compressibility of the fluid.

One should recognize that such studies can elucidate only a small part of the phenomena associated with underwater explosions. An investigation of the migration and oscillation of the explosion bubble requires different analytical methods. Furthermore, if the explosion is too violent the linearized boundary condition on the free surface may not be a useful approximation.

Freely floating bodies. The motion of a freely floating body following an initial displacement is of considerable interest and practical importance, but also leads to a difficult mathematical problem. Uniqueness of solution follows from the argument in section 22 α. For the sake of perspicuity let us restrict ourselves to motion constrained to be vertical, i.e., heaving motion. Then from (19.59) and (19.62) the boundary conditions to be satisfied on the surface of the body in its equilibrium position, S_0, are

$$\phi_n(x,y,z,t) = \dot{y}_1(t)\, n_y(x,y,z)\ ,\ (x,y,z) \text{ on } S_0, \tag{22.61}$$

$$M\ddot{y}_1(t) + \rho g I^A y_1(t) = -\rho \iint_{S_0} \phi_t(\xi,\eta,\zeta,t)\, n_y(\xi,\eta,\zeta)\, d\sigma \tag{22.62}$$

(The notation is explained in section 19 β.) In addition ϕ must satisfy the free-surface condition

$$\phi_{tt}(x,0,z,t) + g\,\phi_y(x,0,z,t) = 0 \tag{22.63}$$

and initial conditions, say

$$\phi_t(x,0,z,0) = \phi_y(x,0,z,0) = 0, \tag{22.64}$$

$$\dot{y}_1(0) = \dot{y}_{10}, \qquad y_1(0) = y_{10}\,. \tag{22.65}$$

As in many previous cases one may reduce the problem to the solution of an integral equation by use of a Green's function. In (13.49) replace (a,b,c) by (ξ,η,ζ) and m(t) by $\gamma(\xi,\eta,\zeta,t)$; denote the resulting function by ϕ_S:

$$\begin{aligned}\phi_S(x,y,z,\xi,\eta,\zeta,t) &= \gamma(\xi,\eta,\zeta,t)\left[\frac{1}{r} - \frac{1}{r_1}\right] \\ &+ 2\int_0^\infty (gk)^{\frac{1}{2}}\, e^{k(y+\eta)}\, J_0(kR)\, dk \int_0^t \gamma(\xi,\eta,\zeta,\tau)\sin\left[(gk)^{\frac{1}{2}}(t-\tau)\right] d\tau.\end{aligned} \tag{22.66}$$

We now attempt to express ϕ by the integral

$$\phi(x,y,z,t) = \iint_{S_0} \phi_S(x,y,z,\xi,\eta,\zeta,t)\, d\sigma, \tag{22.67}$$

for then (22.63) and (22.64) will be satisfied. One should note especially that the relation of ϕ to γ is more complicated here than in problems typified by (16.12), for the past history of γ is involved in ϕ_s . The conditions (22.61) and (22.62) now become

$$-2\pi\gamma(x,y,z,t)+\iint_{S_0}\phi_{sn}(x,y,z,\xi,\eta,\zeta,t)\,d\sigma = \dot{y}_1(t)\,n_y(x,y,z), \tag{22.68}$$

$$M\ddot{y}_1(t)+\rho g I^A y_1(t) = -\rho\iint_{S_0} d\sigma\iint_{S_0} dS\,\phi_{st}(x,y,z,\xi,\eta,\zeta,t)\,n_y(\xi,\eta,\zeta), \tag{22.69}$$

where γ also enters into the equations through ϕ_S . The two equations form a pair of coupled integro-differential equations for γ and y_1 . It is evident that one can probably not hope for an analytic solution even for simple configurations.

Sretenskii [1937b], for two dimensions, and later Haskind [1946b] for three dimensions simplified the problem further by assuming the body to be "thin", i.e., if the surface is given by $z = \pm F(x,y)$, by replacing the boundary condition (22.61) by

$$\phi_z(x,y,\pm 0,t) = \mp\,\dot{y}_1(t)F_y(x,y) \tag{22.70}$$

and S_0 by the projection of S_0 on the plane z = 0 [cf. (20.26) and (20.64)]. With this further assumption one can immediately satisfy (22.68) by taking

$$\gamma(x,y,t) = -\frac{1}{2\pi}\,\dot{y}_1(t)\,F_y(x,y). \tag{22.71}$$

Equation (22.69) then becomes an integro-differential equation for $y_1(t)$.

The procedure is open to some objection in that the substituted boundary condition (22.70) does not seem to fit into the general perturbation scheme as developed in sections 10 α , 19 α and 20 β . It is thus not clear what physical problem really corresponds to the mathematical problem. However, this seems to be the closest anyone has come to reducing the equations to a manageable form. Sretenskii solved his resulting integro-differential equation numerically for a surface described by $F(y) = l e^{\beta |y|}$, where

$$l = \frac{\pi g}{800} = 3.85 \text{ cm}, \quad \beta = \frac{100}{g} = 0.104 \text{ cm}^{-1}.$$

The resulting graph of y_1/y_{10} is shown in Figure 29 with a dimensionless abscissa $t\sqrt{g\beta}$. In spite of the questionableness of the formulation of the problem, the graph is instructive in showing the difference between a damped harmonic oscillation and the solution of Sretenskii's integro-differential equation. Approximate methods of solution to the problem which assume that the fluid motion at any instant is independent of its past history lead to damped harmonic oscillations.

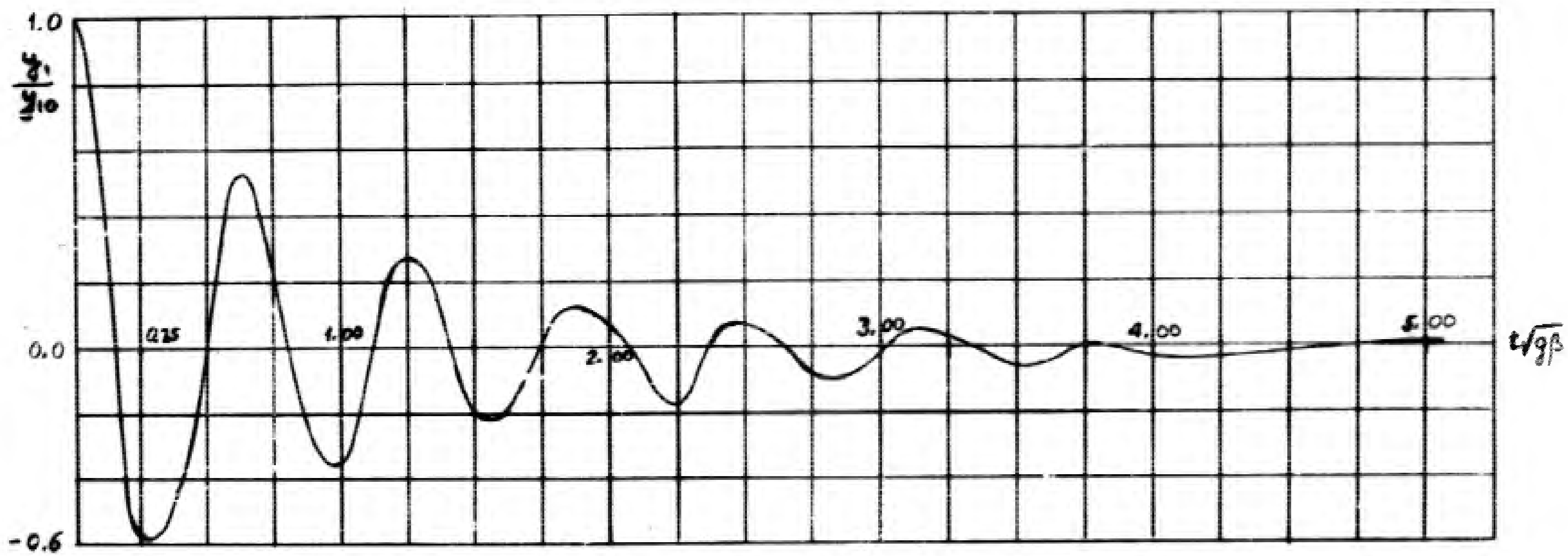

Figure 29

23. Waves in basins of bounded extent

The study of wave motion in a basin presents no special difficulties not already encountered earlier, and has a particular interest because of the many opportunities of observing such waves. Certain general aspects of the problem may be considered as being contained in earlier sections. For example, the general discussion of initial-value problems in section 22α applies to motion in a basin. However, in order to make use of the results, in particular of equation (22.8), in constructing a solution, one must have prior knowledge of the time-dependent Green's function for the geometric boundary. Although the method of images can be used together with (13.49) or (13.53) to construct the Green's function for certain simple configurations, an explicit analytic solution is generally not available.

The time-dependent problem has also been approached in another manner by Hadamard [1910, 1916], who derived an integro-differential equation for the function $\eta(x,y,t)$ describing the free surface. Hadamard's short notes have been worked out by Bouligand [1912, 1926, 1927] and developed further. Certain of Bouligand's investigations indicate that singularities which may occur at the intersection of the plane $y=0$ with the basin walls are a result of linearizing the free-surface boundary condition. For an exact statement one should consult the original papers. There is a brief treatment of Hadamard's equation in Vergne [1928, § 10, 14]. Moiseev [1953] has developed a treatment of the time-dependent problem which generalizes somewhat the method used in section 23α.

In section 23 α we give some general theorems concerning motions periodic in time, and another solution of the initial-value problem. In section 23 β wave motions for several special configurations of the boundary are given. In section 23 γ the theory of wave motion in movable basins is considered.

23 α. Periodic waves in basins—general theorems.

If the motion is periodic in time, the velocity potential may be found by solving a Fredholm integral equation, again after introduction of an appropriate Green's function. Assume $\phi(x,y,z,t) = \varphi(x,y,z)\cos(\sigma t + \tau)$; then φ must satisfy the boundary conditions

$$\begin{aligned} &\varphi_y(x,0,z) - \nu\varphi(x,0,z) = 0, \quad (x,z) \text{ in } F, \quad \nu = \sigma^2/g, \\ &\varphi_n = 0, \quad (x,y,z) \text{ on } S, \end{aligned} \tag{23.1}$$

where F is the part of the plane $y = 0$ occupied by the free surface at rest and S is the surface of the basin. Let $G(x,y,z,\xi,\eta,\zeta)$ be the Green's function for Neumann's problem, i.e.,

$$G = \frac{1}{r} + G_0,$$

where G_0 is regular in the region occupied by fluid and G satisfies the conditions

$$G_n = c \text{ on } S, \quad G_y(x,0,z,\xi,\eta,\zeta) = c \text{ on } F, \tag{23.2}$$

where c is an arbitrary nonzero constant. In addition, in order to make the definition of φ unique we require

$$\iint_{S+F} \varphi \, d\sigma = 0 .$$

It then follows from Green's theorem that

$$\varphi(x,y,z) = \frac{1}{4\pi}\iint_F G\,\varphi_y\,d\xi\,d\zeta = \frac{\nu}{4\pi}\iint_F G\,\varphi(\xi,0,\zeta)\,d\xi\,d\zeta. \tag{23.3}$$

If one now lets $y \to 0$, one obtains

$$\varphi(x,0,z) = \frac{\nu}{2\pi}\iint_F G(x,0,z,\xi,0,\zeta)\,\varphi(\xi,0,\zeta)\,d\xi\,d\zeta, \tag{23.4}$$

a homogeneous Fredholm integral equation for $\varphi(x,0,z)$. If $\varphi(x,0,z)$ can be found, then $\varphi(x,y,z)$ is determined by (23.3). From the theory of such integral equations there will exist a sequence $\nu_1, \nu_2, \ldots$ of eigenvalues for which (23.4) will yield solutions $\varphi_1, \varphi_2, \cdots$. The functions φ_i corresponding to different ν_i's are orthogonal on F, as shown in (16.10). If several ν_i's have the same value, the corresponding φ_i's can be orthogonalized. The φ_i also form a complete set on F. Each solution φ_i yields a standing wave in the basin.

It is possible to use these solutions to solve the initial-value problem formulated in (22.2), but with $p = 0$. Let $\eta(x,z,0)$ and $\eta_t(x,z,0)$ be given. We try to express $\phi(x,0,z,t)$ in the following form:

$$\phi(x,0,z,t) = \sum_{i=1}^{\infty} a_i\,\varphi_i(x,0,z)\cos\sigma_i t + b_i\,\varphi_i(x,0,z)\sin\sigma_i t. \tag{23.5}$$

Then

$$-g\eta(x,z,0) = \phi_t(x,0,z,0) = \sum \sigma_i b_i \varphi_i(x,0,z), \tag{23.6}$$

$$-g\eta_t(x,z,0) = \phi_{tt}(x,0,z,0) = \sum \sigma_i^2 a_i \varphi_i(x,0,z).$$

Since the φ_i form a complete set of orthogonal functions over F, the coefficients a_i and b_i can be determined in the usual manner. $\phi(x,y,z,t)$ is then determined by (23.5) and (23.3).

In order to use the integral equation (23.4) one must first find G, the Green's function to a Neumann problem for a region having a corner along the curve of intersection of the plane $y=0$ and the basin walls. The difficulty with the corner can be overcome in certain cases. If the basin wall intersects the plane perpendicularly, then the basin plus its reflection in the plane $y=0$ has a boundary without this corner. If $\gamma(x,y,z,\xi,\eta,\zeta)$ is a Green's function for the Neumann problem for the extended region, then

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{2}\left[\gamma(x,y,z,\xi,\eta,\zeta)+\gamma(x,-y,z,\xi,\eta,\zeta)\right] \tag{23.7}$$

is a Green's function for the original region. For some other special regions one may construct a Green's function by the method of images, even though the intersection with the plane $y=0$ is not perpendicular.

As mentioned above, each φ_i represents a standing wave of frequency σ_i. It may happen, as we shall see presently, that two or more σ_i's are equal. Let σ be such an eigenvalue and $\varphi^{(1)}$ and $\varphi^{(2)}$ two of the corresponding potential functions. By forming the standing-wave solution

$$\left[\lambda_1\varphi^{(1)}+\lambda_2\varphi^{(2)}\right]\cos\sigma t,\quad \lambda_1+\lambda_2=1, \tag{23.8}$$

one may vary continuously the position of the nodal curves, say. If n independent φ_i correspond to σ, then the possible nodal

curves form an $(n-1)$-parameter family of curves in F. With the two solutions $\varphi^{(1)}$ and $\varphi^{(2)}$ one may also form the solution

$$\phi(x,y,z,t) = \varphi^{(1)}(x,y,z)\cos\sigma t + \varphi^{(2)}(x,y,z)\sin\sigma t. \tag{23.9}$$

The nodal curves will now migrate from those of $\varphi^{(1)}$ to those of $\varphi^{(2)}$, and then on again to those of $\varphi^{(1)}$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ have a common zero at, say, (x_0, z_0), then a nodal curve for ϕ will always pass through (x_0, z_0). Near (x_0, z_0) the waves will appear to progress about (x_0, z_0) like spokes moving about a wheel. There may, of course, be several such centers.

23 β. Some special boundaries.

It is possible to give explicit solutions for standing waves for several particular configurations of the basin. The variety of such configurations, however, is rather small. As a preliminary we note that if the basin has a flat bottom at depth h and if the side walls form a vertical cylinder making a section C with $y = 0$ then, from section 13 α, we have

$$\begin{aligned} &\phi(x,y,z,t) = \varphi(x,z)\cosh m_0(y+h)\cos(\sigma t+\tau), \\ &m_0 \tanh m_0 h - \frac{\sigma^2}{g} = 0, \end{aligned} \tag{23.10}$$

where $\varphi(x,z)$ is a solution of

$$\varphi_{xx} + \varphi_{zz} + m_0^2\varphi = 0 \tag{23.11}$$

satisfying

$$\varphi_n = 0 \quad \text{on } C. \tag{23.12}$$

The boundary condition (23.12) will limit m_0 , and hence σ , to a discrete sequence of eigenvalues

$$m_0^{(1)}, m_0^{(2)}, \dots, \quad \sigma_1, \sigma_2, \dots. \tag{23.13}$$

In a coordinate system in which (23.11) can be separated it is usually possible to find the standing waves in basins whose side walls are constant-coordinate surfaces. These statements will be illustrated below for rectangular and cylindrical coordinates.

In connection with the special cases treated below we call attention to papers by Honda and Matsushita [1913] and Sasaki [1914]. The authors investigated experimentally in a systematic way the various modes of motion in rectangular, triangular, circular, ring-shaped, circular-sectorial and ring-sectorial basins and compared measured with calculated periods. In most cases the agreement is with 2%. Photographs showing the various modes were obtained by sprinkling the surface with aluminum powder and exposing a photographic plate for about one period. The nodes then show up as dots, the rest as streaks. In connection with a study of the excitations of waves in a port, McKnown [1953] has also investigated experimentally the standing waves in circular and square basins; some striking photographs are included. Apté [1957] has studied further the theory of the excitation of standing waves in a square basin and has also given experimental results. Perhaps the first theoretical investigation was by Rayleigh [1876, pp. 272-9]; he compared his predicted frequencies with observations of his own and of Guthrie [1875].

<u>Rectangular basin.</u> Let the basin walls be given by

$$x = 0, \quad x = l, \quad z = 0, \quad z = b, \quad y = -h.$$

Then from (13.6) one may write down immediately the solution

$$\varphi = A \cosh m_o(y+h)\cos\frac{q\pi}{l}x\cos\frac{p\pi}{b}z\cos(\sigma t+\tau),$$
$$m_o^2 = \pi^2\left(\frac{q^2}{l^2}+\frac{p^2}{b^2}\right),\quad \frac{\sigma^2}{g} = m_o\tanh m_o h,\quad p,q = 0,1,2,\ldots. \tag{23.14}$$

Thus the choice of the integers p and q determines m_o and then σ. If the basin is square, i.e., $l = b$, then the same values of m_o and σ may correspond to two different solutions obtained by interchanging p and q, assuming $p \neq q$. However, this may also occur for other rectangular basins if b and l are commensurate.

<u>Circular-cylinder basin.</u> Let the basin have radius $\underline{a}$. Then from (13.8) we find the solutions

$$\varphi = A\cosh m_o(y+h)J_n(m_oR)\cos(n\alpha+\delta)\cos(\sigma t+\tau),\quad n = 0,1,2,\ldots,$$

$$J_n'(m_oa) = 0,\quad \frac{\sigma^2}{g} = m_o\tanh m_o h. \tag{23.15}$$

Thus m_o must be selected so that m_oa is one of the zeros of J_n'; this then determines σ. For $n=0$ the wave crests and nodes lie on concentric circles, the number of such nodal circles depending upon which zero of J_0' is used to determine m_o. If $n \geq 1$, then to the same σ there correspond two independent solutions ($\delta = 0$, $\delta = \frac{1}{2}\pi$), and the remarks made in connection with (23.8) and (23.9) apply.

The standing waves in a basin shaped like a sector of a circle may be obtained from (13.8). If α_o is the angle of the sector, then

$$\varphi = A\cosh m_o(y+h)J_{\frac{n\pi}{\alpha_o}}(m_oR)\cos\frac{n\pi}{\alpha_o}\alpha\cos(\sigma t+\tau),\quad n = 0,1,\ldots, \tag{23.15}$$

$$J_{\frac{n\pi}{\alpha_o}}'(m_oa) = 0,\quad \frac{\tau^2}{g} = m_oh\tanh m_oh.$$

If the basin is ring-shaped, with inner radius b and outer radius a, then from (13.8) one finds (cf. Sano [1913], Campbell [1953]):

$$\phi = A\cosh m_0(y+h)\left[Y_n'(m_0 b)J_n(m_0 R) - J_n'(m_0 b)Y_n(m_0 R)\right]\cos(n\alpha+\delta)\cos(\sigma t+\tau),$$

$$Y_n'(m_0 b)J_n'(m_0 a) - J_n'(m_0 b)Y_n'(m_0 a) = 0, \quad \frac{\sigma^2}{g} = m_0 h \tanh m_0 h, \quad n = 0, 1, \ldots \tag{23.16}$$

Formulas for sectors of a ring may be obtained and are similar to (23.15).

<u>Basins with sloping side-walls.</u> There are very few explicit solutions known when the sides are not vertical. If the basin is a horizontal cylinder bounded at either end by vertical walls at, say, $z = 0$ and $z = \ell$, the theory of progressive waves in canals, developed in section 18 γ, can be carried over with only small changes, namely replacement of $\cos(kz - \sigma t)$ by $\cos kz \cos(\sigma t + \tau)$ where now k is restricted to the values $n\pi/\ell$ and σ correspondingly. Thus (18.39) and (18.43) give the velocity potentials, after the indicated modifications, for various modes of oscillation of a fluid in a basin of triangular section whose sides form an angle of 45° with the horizontal. However, even though these formulas may be used also for the two-dimensional modes, when $k = 0$, they do not give the gravest two-dimensional mode except by a limiting process (described, e.g., in Lamb [1932, p. 443]).

The two-dimensional modes of motion in triangular basins whose sides form an angle $\gamma = m\pi/n$ with the horizontal may also be studied by use of the methods introduced in section 17 β for standing waves on beaches. Indeed, it is apparent that Kirchhoff [1879] considered his investigation of waves on beaches as a preliminary to the

problem at hand. Because his approach is systematic we shall describe it.

In order to use the results of 17γ we take one side as $y = -x \tan\gamma$, i.e., $z = re^{i\gamma}$; let the other side be given by

$$z = 2a - re^{i\gamma}. \tag{23.17}$$

Then the complex potential $f(z)$ must satisfy not only (17.31) and (17.32), but also

$$f(2a - re^{i\gamma}) = \bar{f}(2a - re^{-i\gamma}), \tag{23.18}$$

which, taken with (17.31), implies that

$$f(z) = f\left(z e^{-i4\gamma} + 2a e^{-i2\gamma}(1 - e^{-i2\gamma})\right). \tag{23.19}$$

In order to satisfy (17.31), (17.32) and (23.19) Kirchhoff first takes

$$f(z) = B_h z^h + \dots + B_{h+k} z^{h+k}. \tag{23.20}$$

Substitution in (17.32) yields (with $\beta = \exp -2i\gamma$ as before)

$$1 - \beta^h = 0, \quad B_{n+1} = \frac{-i\nu}{n+1}\,\frac{1+\beta^n}{1-\beta^{n+1}} B_n, \quad 1 + \beta^{h+k} = 0. \tag{23.21}$$

Thus, since $\gamma = m\pi/n$, one must have $h = pn$, $p = 0, 1, \dots$, and $k = \frac{1}{2}n = q$, an integer. If one takes $p = 0$, then (23.20) becomes

$$f(z) = B_0\left\{1 + \sum_{s=1}^{q} (-1)^s \frac{\nu^s}{s!}\,\frac{e^{is\gamma}}{\cos s\gamma} \cot\gamma \dots \cot s\gamma \; z^s\right\}. \tag{23.22}$$

Condition (17.31) requires B_0 to be real. Condition (23.18) or (23.19) remains to be satisfied. The function $f(z)$ in its assumed

form, is apparently overdetermined, and it is possible to show that for $q > 3$ not all conditions can be satisfied. For $q = 2, m = 1$ and $q = 3, m = 1$, (23.18) can be satisfied. The potential functions are as follows:

$\gamma = \pi/4$:

$$f(z) = B_0\left[1-(1+i)\nu z + \tfrac{1}{2} i \nu^2 z^2\right] = \tfrac{1}{2} i B_0 (\nu z - 1 + i)^2 \tag{23.23}$$

$$= B_0\left[(1-\nu x)(1+\nu y) - i\nu(x+y)\left(1-\tfrac{1}{2}\nu(x-y)\right)\right], \quad a = 1/\nu, \quad h = 1/\nu ,$$

$\gamma = \pi/6$: (23.24)

$$f(z) = B_0\left[1-(\sqrt{3}+i)\nu z + \tfrac{1}{2}(1+i\sqrt{3})\nu^2 z^2 - \tfrac{1}{6} i \nu^3 z^3\right]$$

$$= -\tfrac{1}{6} i B_0\left[2 + i(\nu z - \sqrt{3} + i)^3\right]$$

$$= -\tfrac{1}{6} B_0\Big[2 + (\nu y + 1)\left[(\nu y+1)^2 - 3(\nu x - \sqrt{3})^2\right] +$$

$$i(\nu x + \nu y\sqrt{3})(\nu x - \nu y\sqrt{3} - 2\sqrt{3})(\nu x - \sqrt{3})\Big], \quad a = \sqrt{3}/\nu, \quad h = 1/\nu .$$

Here h is the depth of fluid at the deepest point. The surface profile for $\gamma = 45^\circ$ is a straight line, for $\gamma = 30^\circ$ a parabola.

In order to find the higher modes of oscillation Kirchhoff returns to the form (17.33) for $f(z)$. It then follows as before that (17.34) must hold and that n must be even, say $2q$. Now, however, instead of taking $\lambda = 1$ it is left to be determined by (23.19). Substitution of (17.33) into (23.19) gives

$$A_{k+2} = A_k \exp i2\lambda\nu a \beta^{k+1}(1-\beta), \quad k = 0, 1, \ldots, n-3. \tag{23.25}$$

Altogether there are then $n-1+n-2 = 2n-3$ independent equations to determine $A_1, \ldots, A_{n-1}$ and also λ and νa. Again the conditions can be satisfied for $\gamma = \pi/4$ and $\gamma = \pi/6$.

The solutions for $\gamma = \pi/4$ are as follows, where C is an arbitrary real constant

$$
\begin{aligned}
f(z) &= C[\cos\lambda\nu(z-a(1-i)) + \cosh\lambda\nu(z-a(1-i))] \\
\lambda &= \coth\lambda\nu a = -\cot\lambda\nu a\,; \\
f(z) &= C[\cos\lambda\nu(z-a(1-i)) - \cosh\lambda\nu(z-a(1-i))] \\
\lambda &= \tanh\lambda\nu a = \tan\lambda\nu a.
\end{aligned}
\tag{23.23}
$$

The values of λ and ν can easily be determined graphically. For the first set of solutions the values of $\lambda\nu a$ will be slightly more than $3\pi/4$, $7\pi/4$, ..., for the second set slightly less than $7\pi/4$, $11\pi/4$, ... These two sets of solutions correspond, respectively to (18.39) and (18.43) with $k = 0$; the eigenvalues λ may be identified with m_i/ν and n_i/ν, respectively. Kirchhoff and Hansemann [1880] carried out an experimental investigation of the first three antisymmetric modes (equation (23.23) gives the first one); they compare frequencies and positions of maxima and minima. The agreement seems satisfactory, although corrections for surface tension were necessary for the two higher modes.

The solution for $\gamma = 30^\circ$ is the following:

$$
\begin{aligned}
f(z) = C\Big[&\frac{1}{\lambda+1}e^{i\lambda\nu[z-a]} + \frac{1}{\lambda-1}e^{-i\lambda\nu[z-a]} + \\
&\frac{1}{\lambda+1}e^{i\beta^2\lambda\nu[z-a-\beta^2 a]} + \frac{1}{\lambda-1}e^{-i\beta^2\lambda\nu[z-a-\beta^2 a]} \\
&+ \frac{1}{\lambda+1}e^{-i\beta\lambda\nu[z-a-\beta a]} + \frac{1}{\lambda-1}e^{i\beta\lambda\nu[z-a-\beta a]}\Big],
\end{aligned}
\tag{23.27}
$$

where C is an arbitrary real constant, $\beta = \frac{1}{2}(1-i\sqrt{3})$, $\beta^2 = -\bar{\beta} = -\frac{1}{2}(1+i\sqrt{3})$, and the eigenvalues for λ and ν are determined by the equations

$$
\frac{\lambda^2-1}{\lambda} = -\sqrt{3}\cot\lambda\nu a, \quad \frac{\lambda^2+\beta}{\lambda} = -i(1+\beta)\cot\beta\lambda\nu a, \quad \frac{\lambda^2+\bar{\beta}}{\lambda} = +i(1+\bar{\beta})\cot\beta\lambda\nu a. \tag{23.28}
$$

If λ is a solution of (23.28), then also $-\lambda$, $\bar{\lambda}$, $\rho\lambda$ and $\bar{\rho}\lambda$ are solutions. There exists a real solution which may be found from the equations

$$\cosh\sqrt{3}\,\lambda\nu a = 2\sec\lambda\nu a - \cos\lambda\nu a, \quad \lambda = \frac{\sinh\sqrt{3}\,\lambda\nu a - \sqrt{3}\sin\lambda\nu a}{\cosh\sqrt{3}\,\lambda\nu a - \cos\lambda\nu a}. \tag{23.29}$$

The other solutions which may be generated from these do not lead to expressions different from (23.27). The first eigenvalue for $\lambda\nu a$ is a trifle to the right of $3\pi/2$. The form of the free surface corresponding to (23.27) is given by

$$\eta(x,t) = \frac{\sigma}{g} C\left\{\frac{-2}{\lambda^2-1}\cos\lambda\nu(x-a) + \left[\frac{1}{\lambda+1}e^{\frac{1}{2}\sqrt{3}\lambda\nu x} + \frac{1}{\lambda-1}e^{-\frac{1}{2}\sqrt{3}\lambda\nu x}\right]\cos\tfrac{1}{2}\lambda\nu(x-2a)\right.$$
$$\left. + \left[\frac{1}{\lambda+1}e^{-\frac{1}{2}\sqrt{3}\lambda\nu(x-2a)} + \frac{1}{\lambda-1}e^{\frac{1}{2}\sqrt{3}\lambda\nu(x-2a)}\right]\cos\tfrac{1}{2}\lambda\nu x\right\}\sin(\sigma t+\tau). \tag{23.30}$$

Note that (23.24) and (23.27) both give only symmetric modes. Macdonald [1896] states that antisymmetric modes, if they exist, cannot be represented in the assumed forms (23.20) or (17.33).

Vint [1923] has succeeded in finding an infinite number of modes of motion in an inverted four-sided pyramid, each of whose sides makes a 45° angle with the horizontal. We refer to the original paper for the exact formulas.

Additional solutions have been obtained by inverse methods by Sen [1927] and by Storchi [1949, 1952]. Storchi's result, although restricted to two-dimensional motion, is neat. Suppose that the form of the free surface is given as $\eta(x,t) = \eta(x)\sin(\sigma t+\tau) = F'(x)\sin(\sigma t+\tau)$, where $F(x)$ is analytic. Then, since $\eta(x) = \sigma g^{-1}\varphi(x,0)$ and $\varphi_y(x,0) = \nu\varphi(x,0)$,

$$f'(x-i0) = \varphi_x(x,0) - i\varphi_y(x,0) = \varphi_x(x,0) - i\nu\varphi(x,0) = \frac{g}{\sigma}\left[F''(x) - i\nu F'(x)\right]$$

and

$$f(x-i0) = \frac{g}{\sigma}\left[F'(x) - i\nu F(x)\right] + \text{const}$$

We may take the constant as zero and write

$$f(x+iy) = \frac{g}{\sigma}\left[F'(x+iy) - i\nu F(x+iy)\right], \tag{23.31}$$

where $F(z)$ is the analytic function determined by $F(x)$. From this we have

$$\begin{aligned}\varphi(x,y) &= \frac{g}{2\sigma}\left\{F'(x+iy) + F'(x-iy) - i\nu\left[F(x+iy) + F(x-iy)\right]\right\},\\ \psi(x,y) &= \frac{-g}{2\sigma}\left\{i\left[F'(x+iy) + F'(x-iy)\right] - \nu\left[F(x+iy) + F(x-iy)\right]\right\}.\end{aligned} \tag{23.32}$$

Any streamline, defined by ψ = real const., can now be taken as determining a possible basin shape corresponding to the assumed standing-wave profile. Storchi applies the procedure to several special choices of F. An obvious disadvantage of this method, as well as of Sen's, is that only one mode of motion is obtained for a resulting basin shape.

23 γ. Waves in movable basins.

In several preceding sections, especially 19 and 22 γ, we considered the wave motion occurring in the presence of an oscillating body when the fluid is exterior to the body. One may attempt analogous problems when the fluid is situated inside the body. Such problems occur in many situations of practical interest, for example, the sloshing of oil in a partly filled compartment of a tanker and the sloshing of fuel in an airplane or rocket. In

each of these cases interest centers upon the dynamics of the whole system as well as upon the effect upon the walls of the container. A further interest in such problems arises from the interpretation of the experiments on standing waves, referred to earlier, carried out by Honda and Matsushita [1913], Sasaki [1914], and Kirchhoff and Hansemann [1880]. The results were intended for comparison with theoretical prediction of standing waves in fixed basins. The waves were actually generated by oscillating the basin and finding the frequencies at which resonance appeared to occur.

We shall not consider the most general motions of the basin consistent with linearization of the free surface conditions, but shall limit ourselves here to a particular problem with small horizontal oscillations. In section 26 α small vertical oscillations of the basin will be considered. The general problem of motion of a body containing fluid with a free surface has been treated by Moiseev [1953] and Narimanov [1956, 1957]. However, both are primarily concerned with small oscillations. Krein and Moiseev [1957] have also considered certain mathematical aspects of this problem. Okhotsimskii [1957] and Rabinovich [1957] have considered the special case when the fluid is situated in a vertical, or almost vertical cylinder; Narimanov also gives special attention to this case. (Publication of the work of these three authors was apparently delayed: it is stated that, for the most part, it was carried out independently of and prior to Moiseev's papers.) A particular problem, the one discussed below, was treated by Sretenskii [1951] and later by Moiseev [1952a,b, 1953]. Two later papers by Moiseev [1954a,b] apply the theory to engineering problems, especially

ships. Waves resulting from a special type of forced oscillation of a rectangular tank have been studied by Binnie [1941] and Tamiya [1958].

Waves in a basin with elastic restoring force. Consider the configuration shown in Figure 30. The coordinate

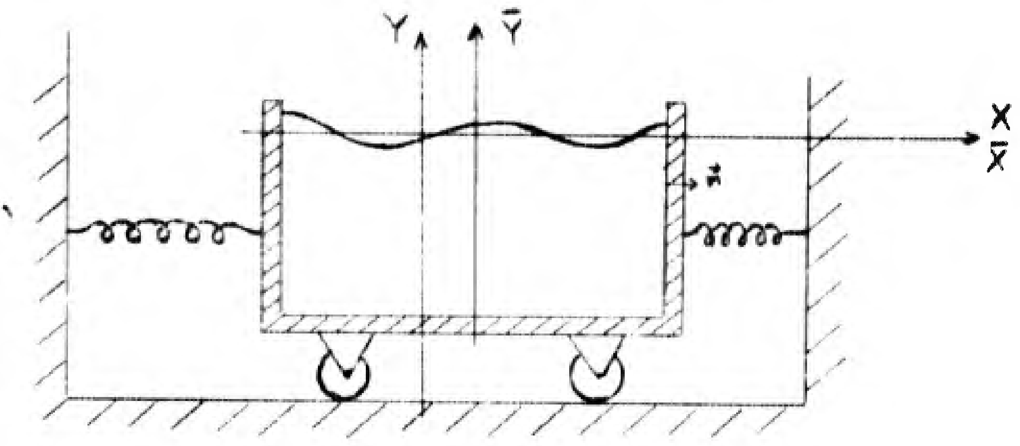

FIGURE 30

system OXY is fixed, the system $\bar{O}\bar{X}\bar{Y}$ moves with the carriage. Let $x_0(t) = O\bar{O}$, $u_0 = \dot{x}_0$. The bottom of the fluid is at $\bar{y} = -h$, the side walls at $\bar{x} = \pm a$. The motion will be taken as two-dimensional. Denote the mass of the carriage, per unit width, by m_c, that of the fluid by m_f and the total by $m = m_c + m_f$. Let the spring constant be $m k^2$. We suppose as usual that the motion may be described by a velocity potential $\phi(x, y, t)$. Following the notation at the end of section 2, let $\bar{\bar{\phi}}(\bar{x}, \bar{y}, t)$ describe the motion relative to the basin, i.e.

$$\phi(x, y, t) = \bar{\bar{\phi}}(\bar{x}, \bar{y}, t) + u_0 \bar{x} \tag{23.33}$$

We shall assume that X_0 and u_0 are both small, and of the same order as $\bar{\bar{\phi}}$, i.e., in the notation of section 10 α, we assume expansions of the form

$$X_0 = \varepsilon X_0^{(1)}, \quad u_0 = \varepsilon u_0^{(1)}, \qquad \text{(23.34)}$$
$$\bar{\bar{\phi}} = \varepsilon \bar{\bar{\phi}}^{(1)} + \varepsilon^2 \bar{\bar{\phi}}^{(2)} + \dots .$$

We omit the formal details of substitution of the perturbation series in the exact boundary conditions. They lead to the following linearized boundary conditions for $\bar{\bar{\phi}}$:

$$\bar{\bar{\phi}}_{tt}(\bar{x},0,t) + g\bar{\bar{\phi}}_{\bar{y}}(\bar{x}_1,0,t) + \ddot{u}_0\bar{x} = 0,$$
$$\bar{\bar{\phi}}_{\bar{x}}(\pm a,\bar{y},t) = 0, \qquad \text{(23.35)}$$
$$\bar{\bar{\phi}}_{\bar{y}}(\bar{x},-h,t) = 0 .$$

The pressure, after discarding higher-order terms, is given by

$$p = -\rho\phi_t = -\rho[\bar{\bar{\phi}}_t + \dot{u}_0\bar{x}]. \qquad \text{(23.36)}$$

The motion of the carriage is determined by the equation

$$m_c\ddot{x}_0 = \int p\cos(n,\bar{x})\,ds - mk^2X_0, \qquad \text{(23.37)}$$

where the integral is taken over the wetted surface when the system is at rest. Substitution of (23.36) gives

$$m\ddot{x}_0 = -\rho\int\bar{\bar{\phi}}_t\,ds - mk^2x_0 \qquad \text{(23.38)}$$
$$= -\rho\int_{-h}^{0}\int_{-a}^{a}\bar{\bar{\phi}}_{tx}\,dx\,dy - mk^2x_0 .$$

(Equation (23.38) is also a direct consequence of conservation of momentum.)

The velocity potential $\bar{\bar{\phi}}$ and the displacement χ_o must be determined together from equations (23.35) and (23.38) and either initial conditions or the further assumption that the motion is harmonic in t.

As a preliminary we shall first suppose that the basin motion, i.e. χ_o , is given, so that only (23.35) need be satisfied. One may try separation of variables and express $\bar{\bar{\phi}}$ in the form

$$\bar{\bar{\phi}} = \sum T_n(t)\, X_n(\bar{x})\, Y_n(\bar{y}). \tag{23.39}$$

Laplace's equation and the last two condition of (23.35) are satisfied by

$$X_{2n} Y_{2n} = \cos\frac{2n}{2a}\pi\bar{x}\cosh\frac{2n\pi}{2a}(\bar{y}+h), \tag{23.40}$$

$$X_{2n+1} Y_{2n+1} = \sin\frac{2n+1}{2a}\pi\bar{x}\cosh\frac{2n+1}{2a}\pi(\bar{y}+h).$$

In order to find the corresponding T_n , expand x in a Fourier series:

$$x = \sum_{n=0}^{\infty}(-1)^n\frac{8a}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2a}\pi x \tag{23.41}$$

and substitute (23.39) and (23.41) in the first condition of (23.35):

$$\sum_{n=0}^{\infty}\left[\ddot{T}_{2n}\cosh\frac{2n}{2a}\pi h + T_{2n}\, g\frac{2n\pi}{2a}\sinh\frac{2n}{2a}\pi h\right]\cos\frac{2n}{2a}\pi x \tag{23.42}$$

$$+\sum_{n=0}^{\infty}\left[\ddot{T}_{2n+1}\cosh\frac{2n+1}{2a}\pi h + T_{2n+1}\, g\frac{2n+1}{2a}\pi\sinh\frac{2n+1}{2a}\pi h + \dddot{x}_o(-1)^n\frac{2a}{(2n+1)^2\pi^2}\right]\sin\frac{2n+1}{2a}\pi x$$

$$= 0$$

Let us set

$$\sigma_n^2 = g\pi\frac{n}{2a}\tanh\frac{n}{2a}\pi h, \quad b_{2n+1} = -(-1)^n\frac{2a}{(2n+1)^2\pi^2}\operatorname{sech}\frac{2n+1}{2a}\pi h. \tag{23.43}$$

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Then equation (23.42) yields the infinite set of differential equations

$$\ddot{T}_{2n} + \sigma_{2n}^2 T_{2n} = 0, \qquad (23.44)$$

$$\ddot{T}_{2n+1} + \sigma_{2n+1}^2 T_{2n+1} = b_{2n+1} \dddot{x}_0 .$$

The solution of the first set, $T_{2n} = A_{2n}\cos(\sigma_{2n}t + \tau_{2n})$, is independent of the motion of the basin and yields the symmetric modes of oscillation in a fixed basin. The solution to the second set may also be found by elementary methods, but will not be given here. However, we note that, if x_0 is harmonic, it confirms that resonance occurs at the frequencies of the asymmetric modes for a fixed basin.

We now turn to the joint solution of (23.35) and (23.38). Substitute (23.39) into (23.38). Then, after evaluating the integral, one finds

$$m\ddot{x}_0 + \frac{4a\rho}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sinh\frac{2n+1}{2a}\pi h\,\dot{T}_{2n+1} + mk^2 x_0 = 0. \qquad (23.45)$$

The equations (23.44) and (23.45) taken together may now be used to determine x_0 and the T_n . If we formulate an initial-value problem by requiring, say,

$$x_0(0) = c_0, \quad \dot{x}_0(0) = 0, \quad \bar{\bar{\phi}}_{\bar{y}}(\bar{x},\bar{y},0) = 0, \quad \bar{\bar{\phi}}_t(\bar{x},\bar{y},0) = 0, \qquad (23.46)$$

then the T_{2n} are all zero and the T_{2n+1} and x_0 must be determined from the differential equations. As usual, one looks for a solution in the form

$$x_0 = c\,e^{-i\omega t}, \quad T_{2n+1} = d_{2n+1}e^{-i\omega t}, \qquad (23.47)$$

where both c and d_{2n+1} may, of course, be complex. Substitution in (23.44) and (23.45), followed by elimination of d_{2n+1}, yields the following equation for determining ω :

$$\omega^2 - k^2 = \frac{32\, a^2 \rho}{\pi^3\, m}\, \omega^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \tanh \frac{2n+1}{2a} \pi h \, \frac{1}{\omega^2 - \sigma_{2n+1}^2} \qquad (23.48)$$

One may find the solutions graphically by plotting each side of the equation as function of ω^2. Figure 31 gives a qualitative idea

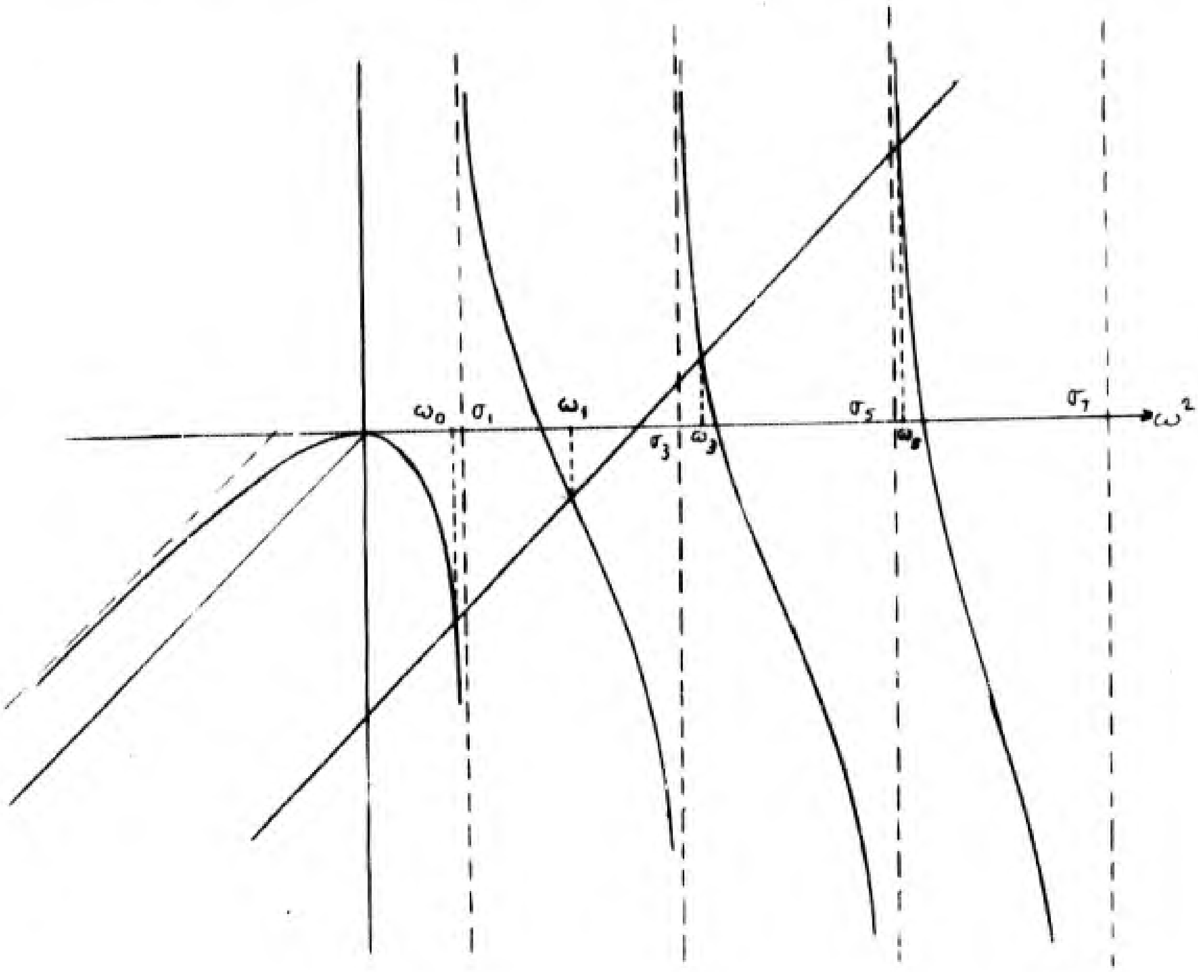


FIGURE 31

of the distribution of solutions $\omega_0, \omega_1, \ldots$. As $n \to \infty$, $\omega^2_{2n+1} - \sigma^2_{2n+1} \to 0$; this fact, which can be proved analytically and which seems clear from Figure 31, would not have been so evident if we had divided (23.48) by ω^4 before plotting. A point of importance is that there is no intersection for $\omega^2 < 0$; as a result the motion is stable. This may be proved as follows. Since $x^{-1}\tanh x \le 1$, the right hand side of (23.48), for $\omega^2 < 0$, is greater than or equal to

$$\frac{32 a^2 \rho}{\pi^3 m} \omega^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \frac{2n+1}{2a} \pi h = \frac{2ah\rho}{m} \omega^2 = \frac{m_f}{m} \omega^2 > \omega^2 . \tag{23.49}$$

Hence the line $\omega^2 - k^2$ lies below the left-hand branch of the curve for $k^2 \geq 0$. The eigenvalues ω_i depend upon the parameters k^2, $2a/h$ and $2\rho ah/m = m_f/m$. Figure 32 from Moiseev [1953] shows the dependence of the fundamental mode ω_0 upon $2a/h$ for two values of m_f/m and $k^2 = 1$.

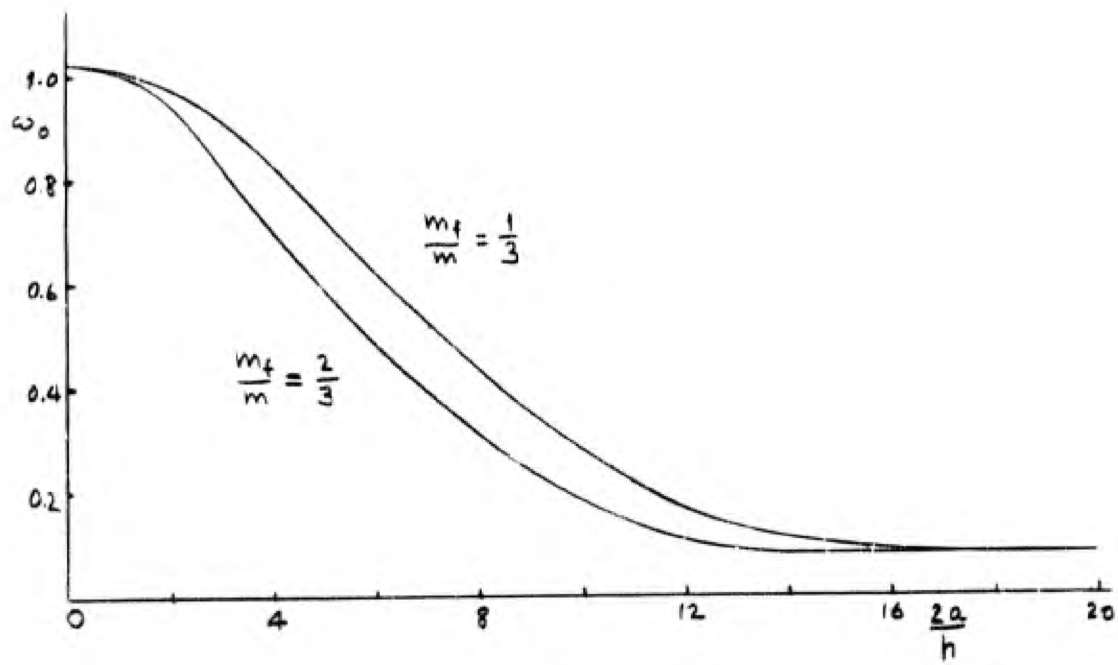


FIGURE 32

The general solution for x_0 and T_{2n+1} is

$$x_0(t) = \mathrm{Re} \sum_{s=0}^{\infty} c_s e^{-i\omega_s t}, \quad T_{2n+1}(t) = \mathrm{Re} \sum_{s=0}^{\infty} d_{2n+1,s} e^{-i\omega_s t}. \tag{23.50}$$

The solution of the initial-value problem formulated in (23.46) will not be completed. It involves solution of infinite sets of linear equations. Approximate solutions can be obtained by considering only a finite number of equations and variables.

The general theory of stability of such systems is discussed in Moiseev's 1953 paper. In an earlier papers [1952b] he studies the special case of a basin containing fluid and serving as the bob of a pendulum. If the suspension is by a parallelogram linkage, so that the container moves parallel to itself, the motion is always stable; if the suspension is by a rod rigidly attached to the container, the motion may be, under certain circumstances, unstable.

The last cited paper by Moiseev describes briefly the results of an experiment with a pendulum; the measured and computed fundamental frequencies for the two systems of suspension agreed with 0.1%.

24. Gravity waves in the presence of surface tension.

Apparently the first one to investigate the theory of waves in a fluid acted upon by both gravity and surface tension was Kelvin [1871a,b]. However, many of the essential features had been discovered earlier through observation by Russell [1844] and others; references may be found in Kelvin's papers. A good account of the classical researches of Kelvin and Rayleigh may be found in Lamb [1932, § 265-272]. Also, Chapter XX of Rayleigh's Theory of Sound [Cambridge, 1929; Dover, N.Y., 1945] contains an exposition of many of his own fundamental researches on surface-tension phenomena, including waves.

The chief mathematical complication added by the action of surface tension is a somewhat more elaborate dynamical boundary condition at an interface or free surface. The difference of primary physical interest lies in the existence of a minimum wave velocity and of two wave lengths with the same velocity. Many of the special problems considered in preceding sections can also be solved when surface tension is acting. However, there has been little motivation for carrying through such analyses for wave motion associated with solid boundaries, since it has been recognized that the additional forces would be small. A further difficulty also appears when the solid boundary pierces the surface, for an additional boundary condition is required at the intersection. As a result, most of the investigations have dealt with waves analogous to those considered in sections 14 α, β and δ, 15, and 22 β. In fact, the complex velocity potentials for the Cauchy-Poisson initial-value problem including the effect of surface tension, has already been

given in equations (22.45) and (22.46). A topic of particular geophysical interest, the stability of an interface, will be dealt with in section 26. Waves on the surface of a viscous fluid, including surface tension, are considered in section 25.

Boundary conditions. The linearized conditions to be satisfied at the interface of two fluids have already been given in equations (10.7) and (10.8) (we recall that subscript 1 refers to the lower fluid). If one eliminates η from these two equations and makes use of the fact that Laplace's equation is satisfied on either side of the boundary, one has the following condition:

$$\Delta \phi_1 = 0 \quad \text{for } y<0, \quad \Delta\phi_2 = 0 \quad \text{for } y>0. \tag{24.1}$$

$$\eta_t(x,z) = \phi_{1y}(x,0,z,t) = \phi_{2y}(x,0,z,t), \tag{24.2}$$

$$\rho_1\left[\phi_{1tt}(x,0,z,t) + g\phi_{1y} + \frac{T}{\rho_1}\phi_{1yyy}\right] = \rho_2\left[\phi_{2tt} + g\phi_{2y}\right] \tag{24.3}$$

If the upper fluid is absent, one sets ρ_2 and ϕ_2 equal to zero and may, of course, drop the subscript.

If the motion is two-dimensional one may introduce a stream function Ψ and a complex potential $F(z,t) = \phi + i\Psi$ and express (24.2) and (24.3) as follows:

$$\eta_t(x) = \operatorname{Im} F_1'(x-i0) = \operatorname{Im} F_2'(x+i0) \tag{24.4}$$

$$\operatorname{Re}\rho_1\left\{F_{1tt}(x-i0) + igF_1' - i\frac{T}{\rho_1}F_1'''\right\} = \operatorname{Re}\rho_2\left\{F_{2tt}(x+i0) + igF_2'\right\} \tag{24.5}$$

If the upper fluid is absent and if the lower fluid is infinitely deep, one may extend the reasoning which led up to Levi-Civita's differential equation (22.25) to derive the following one which must be satisfied for all ζ :

$$F_{tt}(\zeta,t) + igF' - i\frac{T}{\rho}F''' = 0 \tag{24.6}$$

Furthermore, if the fluid is of constant depth h, Cisotti's equation (22.30) may also be extended to include the effect of surface tension:

$$F_{tt}(\zeta+ih,t) + F_{tt}(\zeta-ih) + ig[F'(\zeta+ih) - F'(\zeta-ih)] - i\frac{T}{\rho}[F'''(\zeta+ih) - F'''(\zeta-ih)] = 0$$

$$\text{for } -2h < y < 0. \tag{24.7}$$

Elementary solutions. Let us suppose first that only one fluid is present, and in addition that

$$\phi(x,y,z,t) = \varphi(x,y,z)\cos(\sigma t + \tau).$$

Then φ must be a potential function satisfying

$$-\sigma^2\varphi(x,0,z) + g\varphi_y + \frac{T}{\rho}\varphi_{yyy} = 0. \tag{24.8}$$

Just as in section 13 α, we may separate out the y-variable and obtain the following expressions:

infinite depth:

$$\varphi(x,y,z) = Ae^{my}\varphi(x,y) \tag{24.9}$$

where

$$\Delta_2\varphi + m^2\varphi = 0$$

and

$$\sigma^2 = gm + \frac{T}{\rho}m^3,$$

depth h:

$$\varphi(x,y,z) = A \cosh m_0(y+h)\, \varphi(x,z), \tag{24.10}$$

where

$$\Delta_2 \varphi + m_0^2 \varphi = 0$$

and

$$\sigma^2 = (g m_0 + \frac{T}{\rho} m_0^3) \tanh m_0 h .$$

One may also with no difficulty construct solutions analogous to (13.3) and (13.4), namely

$$\varphi(x,y,z) = A\left[m\left(1 - \frac{T}{\rho g} m^2\right)\cos my + \frac{\sigma^2}{g}\sin my\right]\varphi(x,z) \tag{24.11}$$

and

$$\varphi(x,y,z) = A \cos m_i(y+h)\, \varphi(x,z) \tag{24.12}$$

for infinite and finite depth, respectively, where m_i in (24.12) must satisfy

$$\sigma^2 = (-g m_i + \frac{T}{\rho} m_i^3) \tan m_i h$$

and $\varphi(x,z)$ must be a solution of

$$\Delta_2 \varphi - m^2 \varphi = 0.$$

Unfortunately, the set of function

$$\left\{\cosh m_0(y+h),\ \cos m_i(y+h)\right\}$$

is no longer orthogonal in general, so that the convenience of general solutions like (16.3) is lost.

It is not necessary to repeat the computations of section 13 since they remain unaltered. Essentially the only change is in the relation between the frequency σ and the wave number m. Here the fact of predominant physical interest is that for small values of m the relation is controlled chiefly by the gravitational con-

stant g and for large values of m by T/ρ.

If one forms two-dimensional progressive waves by superposing the standing-wave solutions obtained from (24.9) and (24.10), a further significant physical fact appears: the wave velocity now has a minimum for some value of $m > 0$, except for very shallow depth. These facts are displayed graphically in Figure 12 and further information is given in the associated discussion (the curves were computed for water at 20°C and $h = \infty$ or 1 cm). Formulas for the position of the minimum and various associated values are given for infinite depth in the following table; the numerical values are for water at 20°C (T = 72.8 dynes/cm, ρ = .998 gm/cm^3):

$$m_m = \sqrt{\rho g/T} = 3.66 \text{ cm}^{-1}$$

$$\lambda_m = 2\pi\sqrt{T/\rho g} = 1.71 \text{ cm}$$

$$c_m = \sqrt[4]{4gT/\rho} = 23.1 \text{ cm/sec}$$

$$\sigma_m = \sqrt[4]{4\rho g^3/T} = 84.8 \text{ radians/sec} = 13.5 \text{ cycles/sec} \qquad (24.13)$$

When $h \leq \sqrt{3T/2\rho g}$ there is no longer a minimum value of c for $m > 0$; in this case c increases monotonically with m. The critical depth for water is about .33 cm. Except in this latter case every value of c has associated with it waves of two different lengths, each of which travels with velocity c. Kelvin suggested that the shorter waves, whose behavior is controlled chiefly by surface tension be called "ripples". The suggestion has been followed for the most part [French: "rideaux"; German: "Rippeln" or "Kräuselwellen"; Russian: "ryaby"], but they are frequently also called "capillary waves" in contrast with "gravity waves".

The relation between σ and m was subjected to a rather thorough experimental investigation by Matthiessen [1889]. He made measurements with water, mercury, alcohol, ethyl ether and carbon disulfide with frequencies ranging from 8.4 to roughly 2000 cycles per second. Agreement between theory and measurement is generally within 5% with the greatest discrepancies occurring near the minima. Rayleigh [1890] and Michie Smith [1890] were apparently the first to use the theoretical relation as a means of experimental determination of T, and it has become one of the standard experimental procedures. For more recent developments and further references see Brown [1936] and Tyler [1941].

Solutions for standing or progressive interfacial waves, analogous to those considered in section 14 δ, can be found by application of the same methods. Since the analysis is similar we give only the relation between σ and m. If the two fluids fill the whole space, with their interface at $y = 0$, then

$$\sigma^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g m + \frac{T}{\rho_1 + \rho_2} m^3 \tag{24.14}$$

If the lower fluid is of depth d_1, the upper of depth d_2 and the interface at $y = 0$, then

$$\sigma^2 = \frac{(\rho_1 - \rho_2) m g + T m^3}{\rho_1 \coth m d_1 + \rho_2 \coth m d_2} \tag{24.15}$$

In both (24.14) and (24.15) $\sigma^2 > 0$ if

$$\rho_2 < \rho_1 + \frac{T m^2}{g}; \tag{24.16}$$

thus the motion may be stable even when the lower fluid is less dense than the upper one. This is not true in the absence of surface tension, as inspection of (14.29) and (14.30) shows.

The analogue of the next example of section 14 δ is somewhat more complex, for two surface tensions are necessary. Let T be the surface tension at the free surface $y=0$, and T_{12} that at the interface $y=-d_1$: let the rigid bottom be at $y=-h=-d_1-d_2$. Then the relation analogous to (14.31) is

$$\begin{aligned}\left(\frac{\sigma^2}{gm}\right)^2\left[\rho_1 \coth m d_1 \coth m d_2+\rho_2\right] \qquad (24.17)\\ -\frac{\sigma^2}{gm}\rho_1\left[\left(1+\frac{T+T_{12}}{\rho_1 g}m^2\right)\coth m d_2+\rho_1\left(1+\frac{T}{\rho_2 g}m^2\right)\coth m d_1\right]\\ +\left(1+\frac{T}{\rho_2 g}m^2\right)\left[\rho_1\left(1+\frac{T+T_{12}}{\rho_1 g}m^2\right)-\rho_2\left(1+\frac{T}{\rho_2 g}m^2\right)\right]=0.\end{aligned}$$

The assumption of $d_1=\infty$ no longer results in any notable simplification of the equation. However, one may show that the solutions σ^2/gm are always real, and that they are positive if

$$\rho_2<\rho_1+\frac{T_{12}}{g}m^2. \qquad (24.18)$$

This is the same condition for stability as was found in (24.16) (and is still necessary as well as sufficient). Much of the rest of the pure gravity-wave analysis of section 14 δ may be carried through. Thus, if σ is the larger and σ_2 the smaller root of (24.17) for a given m, then one may establish the inequality

$$\begin{aligned}0<\frac{\sigma_2^2}{gm}<\left(1+\frac{T}{\rho_2 g}m^2\right)\tanh m d_2<\frac{\sigma_1^2}{gm}< \qquad (24.19)\\ <\left(1+\frac{T}{\rho_2 g}m^2\right)\left(1+\frac{T+T_{12}}{\rho_1 g}m^2\right)\min\left\{1,\frac{\rho_1}{\rho_2}\tanh mh\right\}.\end{aligned}$$

If η and η_{12} are the profiles of the free surface and interface, respectively, then one finds, analogously to (14.34),

$$\frac{\eta_{12}}{\eta} = \cosh m d_2 - \frac{gm}{\sigma_2}\left(1+\frac{T}{\rho_2 g}m^2\right)\sinh m d_2 ; \tag{24.20}$$

again, it follows from (24.18) that this ratio is positive for the larger and negative for the smaller of the two roots of (24.17). The discussion of the nature of the motion associated with the root σ_2 may be taken directly from section 14 δ ; however, the upper bound for the velocity c_2 of a progressive wave of wave number m is now given by

$$c_2^2 = \frac{\sigma_2^2}{m^2} = \frac{\sigma_2^2}{gm}\,\frac{g}{m} < \frac{g}{m}\left(1+\frac{T}{\rho_2 g}m^2\right)\tanh m d_2 < g d_2\left(1+\frac{T}{\rho_2 g}m^2\right). \tag{24.21}$$

Let us turn next to the situation in which the two fluids are moving and look for possible steady motions. Assume each fluid to move to the left with mean velocity c_i and let $F_i(z)$, $i = 1, 2$, be the complex velocity potentials. We again look for solutions in the form [cf. (14.36)]

$$F_i(z) = -c_i z + f_i(z), \quad i = 1, 2, \tag{24.22}$$

where f_i is assumed small with respect to $c_i z$. Then the linearized boundary conditions corresponding to (14.37) are

$$\eta(x) = \frac{1}{c_1}\operatorname{Im} f_1(x-i0) = \frac{1}{c_2}\operatorname{Im} f_2(x+i0), \tag{24.23}$$

$$\frac{\rho_2}{c_2}\operatorname{Re}\left\{igf_2(x+i0) + c_2^2 f_2'(x+i0)\right\} = \frac{\rho_1}{c_1}\operatorname{Re}\left\{igf_1(x-i0) + c_1^2 f_1'(x-i0) - i\frac{T}{\rho_1}f_1''(x-i0)\right\}.$$

If we look for a steady motion of the form

$$f_1 = a_1 e^{-imz}, \quad f_2 = a_2 e^{-imz}, \tag{24.24}$$

then substitution in (24.23) yields

$$\frac{a_1}{c_1} = -\frac{\bar{a}_2}{c_2}$$

and

$$m(\rho_1 c_1^2 + \rho_2 c_2^2) = (\rho_1 - \rho_2)g + Tm^2 \tag{24.25}$$

The last equation will not have a real solution for m, assuming $\rho_1 > \rho_2$, unless

$$4g(\rho_1 - \rho_2)T \leq (\rho_1 c_1^2 + \rho_2 c_2^2)^2 . \tag{24.26}$$

There are then two solutions of the form (24.24). The effect of surface tension may be seen more clearly if one graphs each side of (24.25) and finds the intersections, if any. It will be shown in section 26 that this type of motion is unstable if $|c_1 - c_2|$ becomes too large.

Singular solutions. The methods used in section 13 γ for finding source-type solutions can generall; be extended to take account of surface tension. Aside from the algebraic complications the chief difficulties are associated with selecting the proper boundary conditions at infinity. For a stationary source of pulsating strength one may still impose a radiation condition as in (13.9) and obtain the correct solution. However, for the steadily moving source of constant strength the proper choice is no longer clear. Although it is possible to fall back upon arguments based upon considerations of group velocity, they are

not completely convincing, so that it seems safer to formulate first an initial-value problem which can yield either of the two cases mentioned above as a limit when $t \to \infty$. First we give the velocity potential for a source of variable strength $m(t)$, $t \geq 0$, moving on an arbitrary path $(a(t), b(t), c(t))$. The potential function ϕ must satisfy the same conditions given on p. 90 except that 2) is now replaced by

$$\phi_{tt}(x,0,z,t) + g\phi_y(x,0,z,t) + \frac{T}{\rho}\phi_{yyy}(x,0,z,t) = 0. \tag{24.27}$$

There is no special difficulty involved in finding ϕ. For infinite depth, it is as follows:

$$\phi(x,y,z,t) = \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} + \tag{24.28}$$
$$2\int_0^\infty dk\sqrt{gk+T'k^3}\int_0^t d\tau\, m(\tau)\sin\left[(t-\tau)\sqrt{gk+T'k^3}\,\right]e^{k\left(y+b(\tau)\right)}J_0(kR(\tau)),$$

where we have written T' for T/ρ. One may similarly find the function analogous to (13.53) by replacing gk by $gk + T'k^3$. Knowledge of these functions allows one now to repeat, at least in part, the considerations of sections 22α and 22β.

For a stationary source at (a, b, c) with strength $m\cos\sigma t$, the velocity potential may be easily derived from (24.28). It is as follows:

$$\phi(x,y,z,t) = \left[\frac{1}{r} + \frac{1}{r_1} + 2\sigma^2\int_0^\infty \frac{1}{T'k^3+gk-\sigma^2}\, e^{k(y+b)}J_0(kR)\,dk\right] m\cos\sigma t$$
$$+ 2\pi m\,\frac{\sigma^2}{g+3T'k_0^2}\, e^{k_0(y+b)}J_0(k_0R)\sin\sigma t, \tag{24.29}$$

where k_0 is the real solution of $\sigma^2 = gk + T'k^3$. If the fluid is of depth h, then

$$\phi(x,y,z,t) = \Big[\frac{1}{r} + \frac{1}{r_2} + \tag{24.30}$$

$$2\int_0^\infty \frac{T'k^3+gk+\sigma^2}{T'k^3+gk-\sigma^2\coth kh}\;\frac{e^{kh}\cosh k(b+h)\cosh k(y+h)}{\sinh kh}\;J_0(kR)\,dk\Big]m\cos\sigma t$$

$$+2\pi m\,\frac{T'k_0^3+gk_0+\sigma^2}{\sigma^2 h+(3T'k_0^2+g)\sinh^2 k_0 h}\;e^{k_0 h}\sinh k_0 h\cosh k_0(b+h)\cosh k_0(y+h)\,.\,J_0(k_0R)\sin\sigma t,$$

where k_0 is the real root of $T'k^3+gk-\sigma^2\coth kh=0$.

The velocity potential for a source moving in the direction OX with constant velocity u_0 may also be obtained from (24.38) by a suitable limiting procedure, although the computation is somewhat more tedious. In a coordinate system moving with the source it is as follows for $h=\infty$:

$$\varphi(x,y,z) = \frac{m}{r} - \frac{m}{r_1} \tag{24.31}$$

$$+\frac{4m}{\pi}\int_0^{\frac{1}{2}\pi} d\vartheta\;\mathrm{PV}\!\!\int_0^\infty dk\,\frac{g+T'k^2}{g+T'k^2-ku_0^2\cos^2\vartheta}\,e^{k(y+b)}\cos[k(x-a)\cos\vartheta]\cos[k(z-c)\sin\vartheta]$$

$$+4m\sum_{i=1,2}(-1)^{i-1}\int_0^{\vartheta_0} d\vartheta\,k_i(\vartheta)\,\frac{T'k_i^2+g}{T'k_i^2-g}\,e^{k_i(y+b)}\sin[k_i(x-a)\cos\vartheta]\cos[k_i(z-c)\sin\vartheta],$$

where

$$\vartheta_0 = \begin{cases} \operatorname{Arc\,cos}(4gT')^{1/4}/u_0 & \text{if } 4gT' \le u_0^4 \\ 0 & \text{if } 4gT' \ge u_0^4 \end{cases}$$

and

$$\left.\begin{matrix} k_1(\vartheta) \\ k_2(\vartheta)\end{matrix}\right\} = \frac{u_0^2\cos^2\vartheta \pm \sqrt{u_0^4\cos^4\vartheta - 4gT'}}{2T'}\,.$$

One may easily show that

$$\nu \sec^2\vartheta < k_1(\vartheta) \le \sqrt{g/T'} \le k_2(\vartheta) \le u_0^2 \cos^2\vartheta / T'.$$

As $T' \to 0$ it is then evident that the integral involving k_2 vanishes and that (24.31) reduces to (13.36).

One may carry out an asymptotic investigation of (24.31), or of φ_x, along the lines of (13.36) and following. However, the analysis is considerably more complicated. The behavior of the wave pattern is roughly as follows. For $u_0^4/4gT' \le 1$, $\varphi_x(R, \alpha, y)$ is $O(R^{-1})$ for all α, and the disturbance is chiefly local. There is a constant $c > 1$ such that when $1 < u_0^4/4gT' < c$ the wave pattern is a superposition of two sets of waves corresponding to the two roots k_1 and k_2. Those corresponding to k_2 are capillary waves which precede the source and bend around it so that their crests eventually make an angle $\frac{1}{2}\pi + \vartheta_0$ with the x-axis. The gravity waves corresponding to k_1 behave similarly except that they follow the source and are longer. If $u_0^4/4gT' > c$, a second angle, say ϑ_1, appears, where $\vartheta_1 < \vartheta_0$. There are now three sets of waves. Those associated with k_2 behave as described above. The gravity waves, however, consist of both transverse waves spanning the angle between $\pm(\frac{1}{2}\pi + \vartheta_1)$ and diverging waves which now lie in the wedge bounded by $\frac{1}{2}\pi + \vartheta_1$ and $\frac{1}{2}\pi + \vartheta_0$ and its reflection. One will find a sketch in Lamb's Hydrodynamics [1932, p. 470] which was computed for the similar problem of a moving pressure point, the so-called "fishline problem". A precise value for the constant c does not seem to be known. The free surface η may be computed from

$$\eta(x, z) = \frac{u_0}{g}\left[\varphi_x(x, 0, z) + \frac{T}{\rho u_0^2}\varphi_{xy}(x, 0, z)\right].$$

In spite of the general complexity of the asymototic analysis of (24.31), it is relatively easy to find the asymptotic form of η directly ahead ($\alpha = 0^{\circ}$) and directly behind ($\alpha = 180^{\circ}$):

$\alpha = 0$: (24.32)

$$\eta(x,z) = -8m \frac{u_0}{g} k_2^2 \left[1 + \frac{T'}{u_0^2} k_2\right] \sqrt{\frac{\pi}{2k_2 R}} \frac{T' k_2^2 + g}{[(T'k_2^2 - g)(3T'k_2^2 + g)]^{1/2}} e^{k_2 b} \cos(k_2 R - \tfrac{1}{4}\pi) + O(R^{-1});$$

$\alpha = 180^{\circ}$: (24.33)

$$\eta(x,z) = 8m \frac{u_0}{g} k_1^2 \left[1 + \frac{T'}{u_0^2} k_1\right] \sqrt{\frac{\pi}{2k_1 R}} \frac{T' k_1^2 + g}{[(g - T'k_1^2)(3T'k_1^2 + g)]^{1/2}} e^{k_1 b} \cos(k_1 R - \tfrac{3}{4}\pi) + O(R^{-1});$$

here

$$k_1 = k_1(0) = \frac{u_0^2}{2T'}\left[1 - \sqrt{1 - 4T'g/u_0^4}\right], \quad k_2 = k_2(0) = \frac{u_0^2}{2T'}\left[1 + \sqrt{1 - 4T'g/u_0^4}\right]$$

and we assume $u_0^4 > 4T'g$. One may see rather clearly the effect upon k_1 and k_2 of varying T' and u_0 by finding them as the intersection of the graphs of $T'k^2 + g$ and $u_0^2 k$.

There is no special difficulty in finding source solutions for two-dimensional motion, and the asymptotic behavior is of course easier to determine. The related problem of a moving concentrated pressure is treated in Lamb [1932, §§ 270,1]. For this problem a paper by DePrima and Wu [1957] is particularly instructive, for they obtain the solution by first formulating the initial-value problem and then finding the limit as $t \to \infty$. In addition, they analyze the form of the surface for large but finite values of t.

25. Waves in a viscous fluid.

If one abandons the assumption of a perfect fluid with irrotational motion, one loses at the same time many convenient and powerful mathematical tools from potential theory and the theory of functions of a complex variable. However, the simplifications introduced by the infinitesimal-wave approximation are sufficient to allow obtaining a number of solutions of interest, most of which have been known for many years. However, discovery of errors in early work has resulted in several recent papers. Furthermore, in connection with the theory of stability of interfaces the subject has again attracted attention; this work will be summarized in section 26. One will find general expositions of many of the fundamental results in Lamb [1932, §§348-351], and Levich [1952, pp. 467-497]. Longuet-Higgins [1953b] gives a valuable discussion of the perturbation procedure and carries through certain second-order computations.

Subject to the limitations of the approximation one can find solutions for periodic standing waves in fluid of both infinite and finite depth with a free surface, at the interface of two different fluids in which either may have a fixed horizontal plane as its other boundary, and at the interface and free surface when two different fluids are superposed, the upper one having a free surface. In all cases the presence of surface tension may be admitted. By making use of such solutions together with Fourier analysis one can find the solution to the Cauchy-Poisson initial-value problem (cf. Sretenskii [1941]).

In general, in the investigation of standing waves one is particularly interested in two things, the effect of viscosity upon the relation between wave-length and frequency, and the rate of decay of amplitude. As an alternative to examining the rate of decay, one may instead assume that a space-and time-periodic pressure has been applied to the free surface and determine the rate of transfer of energy necessary to maintain a steady oscillation.

One may still, as for perfect fluids, combine standing-wave solutions which are out of phase in order to form progressive waves. In a coordinate system moving with the waves the wave system will be stationary but the motion will not be steady for, as a result of viscosity, it will decay unless a periodic pressure distribution is moving with the waves and doing work upon the fluid. Fourier analysis may be used to obtain the fluid motion resulting from an arbitrary moving pressure distribution. Indeed, one need not restrict oneself to a pressure distribution but may also include a distribution of shearing stress at the free surface. If a pressure and shearing distribution of localized extent is moving over the fluid the dissipation of wave energy in viscosity will show up in a diminution of amplitude, as one moves away from the pressure area, which is more rapid than for a perfect fluid. Such problems have been investigated by Sretenskii [1941, 1957] and by Wu and Messick [1958]. The latter include the effect of surface tension and make a particularly thorough study of the behavior of the solution; they restrict themselves to two-dimensional motion. One should keep in mind that if the fluid is of finite depth it is

no longer equivalent to formulate a problem in which the pressure distribution is fixed and the fluid moves with a constant mean velocity.

Instead of attempting to construct a steady progressive-wave solution by means of a moving pressure distribution, one may instead assume that the progressive waves have been somehow initiated and then study their rate of decay with distance from the wave-maker. (This is, of course, closely related to finding the decay with time of an initially given progressive wave.) Studies of this nature have been made by Biesel [1949] and Carry [1956], who investigated especially the effect of the bottom, by Ursell [1952], who investigated the effect of side walls for infinite depth, and by Hunt [1952], who combined the two. Dissipation with distance when no walls are present has been treated by Dmitriev [1953] in connection with the theory of the wave-maker. A point of physical interest in these studies is the relative contribution to dissipation of shearing motion near the surface, near the bottom, near the walls, and within the fluid. Case and Parkinson [1957] have studied the damping of standing waves in a circular cylinder of finite depth, making use of the linearized equations of this section; their experimental data seem to confirm the theoretical predictions when the cylinder walls are sufficiently polished.

The fluid motion resulting from a submerged stationary source of pulsing strength has been derived by Dmitriev [1953] for two-dimensional motion and infinite depth. Sretenskii [1957] has carried through the calculations for steady motion of a source in three dimensions. Unfortunately, the source function is not

now as useful a tool for constructing solutions to special boundary-value problems as it is for perfect fluids. In particular, one can no longer satisfy the proper boundary condition on a steadily moving body by means of distributions of sources and sinks, as was possible in section 20 β. On the other hand, distributions of pulsating sources may still be used to satisfy the linearized boundary conditions on certain types of stationary oscillating bodies. Thus, if the motion is such that the linearized boundary condition specifies the velocity normal to the surface together with zero tangential velocity, then a source distribution may prove useful. For example, the wave-maker problems formulated in (19.26) and (19.31) may be treated in this fashion; Dmitriev [1953] has done this.

A fundamental assumption of the preceding remarks is that the motion is laminar. Such an assumption seems to be in harmony with the assumption of small motions which is made in deriving the equations of the present section. However, the possible occurrence of turbulent motion in progressive waves has been reported by Dmitriev and Bonchkovskaya [1953] who found experimental evidence for it near the surface, where the vorticity was highest. The photographs in Figure 8 do not seem to show any evidence of it, but this may result from special circumstances of the experiments. Bowden [1950] has essayed a theory based on von Kármán's similarity hypothesis; further references are given there. In the case of steady free-surface flow in a channel the importance of turbulence in modifying the mean-velocity profile is almost obvious. However, investigations have been confined to the necessary modifications of the

shallow-water approximation and will be discussed elsewhere.

Finally, we note that much of the theory given below for a constant surface tension T can, in fact, be extended to a more general surface condition. This is indicated in Lamb [1932, § 351] and carried out by Dorrestein [1951] in some detail for infinite depth. He includes compressibility of the surface film, hysteresis and a "surface viscosity".

25 α . Linearized equations and simple solutions.

The linearized equations and boundary conditions have already been derived in section 10. For a stratified fluid with interface at $y=0$ the zeroth-order equations are given in (10.2), the first-order in (10.3). For a single fluid with free surface they are given in (10.4). It is customary and convenient to combine the zeroth- and first-order equations. Thus, if in (10.4) we let $p = p^{(0)} + \varepsilon p^{(1)}$ and $\underline{v} = \varepsilon \underline{v}^{(1)}$, then the equations become

$$u_x + v_y + w_z = 0$$

$$\underline{v}_t = -\frac{1}{\rho}\, \mathrm{grad}\,(p + \rho g y) + \nu \Delta \underline{v}\,, \qquad (25.1)$$

$$u_y + v_x = w_y + v_z = 0 \qquad \text{for } y = 0,$$

$$p - \rho g \eta - 2\mu v_y = -T(\eta_{xx} + \eta_{zz}) + \bar{p} \qquad \text{for } y = 0,$$

$$\eta_t(x, z, t) = v(x, 0, z, t).$$

One may clearly combine (10.2) and (10.3) in the same way. In order to obtain the proper equations in a coordinate system moving to the right with velocity u_0 , one need only replace $\partial/\partial t$ by $\partial/\partial t - u_0 \partial/\partial x$.

The standard procedure for solving the equations is to represent the motion as a potential flow plus a rotational flow and to determine the pressure from the potential part. Thus, let

$$\underset{=}{v} = \underset{=}{v}^{(p)} + \underset{=}{v}^{(r)} \tag{25.2}$$

where

$$\underset{=}{v}^{(p)} = \operatorname{grad} \phi \tag{25.3}$$

and let

$$p = -\rho \phi_t - \rho g y. \tag{25.4}$$

It then follows from the second equation in (25.1) that $\underset{=}{v}^{(r)}$ must satisfy

$$\frac{\partial}{\partial t} \underset{=}{v}^{(r)} = \nu \Delta \underset{=}{v}^{(r)} \tag{25.5}$$

The relation between $\underset{=}{v}^{(p)}$ and $\underset{=}{v}^{(r)}$ is established through the boundary conditions. In the several examples treated below the motion is two-dimensional. However, there is no difficulty in principle and not much additional algebraic complexity in solving the analogous three-dimensional problems. The essential simplification in two dimensions is that the components of $\underset{=}{v}^{(r)}$ may be expressed, as a consequence of the continuity equation, in terms of a single function Ψ :

$$u^{(r)} = \frac{\partial \Psi}{\partial y}, \quad v^{(r)} = -\frac{\partial \Psi}{\partial x}. \tag{25.6}$$

It then follows easily from (25.5) that

$$\frac{\partial \Psi}{\partial t} = \nu \Delta \Psi. \tag{25.7}$$

Standing waves-infinite depth. We shall try to find a solution to the equations which has a profile of the form

$$\eta(x,t) = A(t)\cos(mx+\alpha). \tag{25.8}$$

If such a solution exists, the nature of $A(t)$ will, of course, be of especial interest. We take ϕ and Ψ of the form

$$\phi = F(y,t)\cos(mx+\alpha), \quad \Psi = G(y,t)\sin(mx+\alpha). \tag{25.9}$$

Equation (25.7) then implies that

$$\Psi = (ce^{ly} + de^{-ly})e^{\omega t}\sin(mx+\alpha), \tag{25.10}$$

where

$$l^2 = m^2 + \frac{\omega}{\nu}. \tag{25.11}$$

Neither l nor ω need be real. The form of ϕ is further determined by $\Delta\phi = 0$ and its relation to Ψ through the third boundary condition in (25.1). It must be

$$\phi = (ae^{my} + be^{-my})e^{\omega t}\cos(mx+\alpha). \tag{25.12}$$

If, as usual, we require the motion to remain bounded as $y \to -\infty$, we must take $b = 0$. If l has a non-vanishing real part, which we assume for the present, we may without loss of generality take it to be positive. Hence one must have $d = 0$. It follows from the third condition of (25.1) that

$$a = c\,\frac{l^2+m^2}{2m^2}. \tag{25.13}$$

Substitution in the formula for η_t and integration with respect to t yield

$$\eta = c\,\frac{1}{2\nu m}\,e^{\omega t}\cos(mx+\alpha) = A_{\circ}e^{\omega t}\cos(mx+\alpha). \quad (25.14)$$

Finally, one must substitute into the dynamical boundary condition in (25.1). There p is computed from (25.4) with $y = 0$. For future use we retain the external pressure distribution $\bar{p}$, which we take in the form

$$\bar{p} = p_{\circ}\,e^{\omega t}\cos(mx+\alpha), \quad (25.15)$$

where $p_{\circ}$ may be complex. The boundary condition yields an equation relating ℓ and m:

$$\nu^2(\ell^2+m^2)^2 - 4\nu^2m^3\ell + gm + T'm^3 = -m\,\frac{p_{\circ}}{\rho}\,\frac{2m\nu}{c} = -m\,\frac{p_{\circ}}{\rho A_{\circ}}, \quad (25.16)$$

or, by making use of (25.11), an equation relating ω and m:

$$(\omega + 2m^2\nu)^2 - 4\nu^2m^3\sqrt{m^2+\frac{\omega}{\nu}} + gm + T'm^3 = -m\,\frac{p_{\circ}}{\rho A_{\circ}}. \quad (25.17)$$

Consider first equation (25.16) with $p_0 = 0$ and let

$$\zeta = \frac{\ell}{m}, \quad K = \frac{gm + T'm^3}{\nu^2m^4}. \quad (25.18)$$

Then (25.16) takes the dimensionless form

$$(\zeta^2+1)^2 - 4\zeta + K = 0.$$

An examination of this equation shows that two of its roots are always complex with negative real parts. These roots are discarded since the corresponding motion would not die out as $y \to -\infty$; in fact, we explicitly assumed earlier that ℓ has a positive real part. (Note that if we had made the other possible assumption,

i.e., that ℓ had a negative real part, the resulting equation corresponding to (25.16) would have had roots with positive real part, again to be discarded.) The other two roots have positive real part. Whether or not there is an imaginary part depends upon the value of K. There is a critical value $K_c \doteq .581$ such that if $K < K_c$ the two allowable solutions are both real. If $K > K_c$, the solutions are complex conjugates. Let the two complex roots of positive real part be denoted by $\ell_1 \pm i\ell_2$. Then one may establish that $\ell_1/m > .683$. When the two admissible roots are real, both of them lie between 0 and m.

One may find the values of ω associated with the two admissible roots from (25.11). If they are both real $(K < K_c)$, then $\omega = -\nu(m^2 - \ell^2) < 0$. In this case the motion is critically damped and the initial configuration of the surface gradually subsides. This occurs for a given m if ν is large enough. On the other hand, no matter how small ν is, it also occurs when m is large enough, i.e., for very small wave length. If the two admissible roots are complex $(K > K_c)$, then

$$\omega = -\nu m^2\left(1 - \frac{\ell_1^2}{m^2} + \frac{\ell_2^2}{m^2} \pm 2i\frac{\ell_1\ell_2}{m^2}\right)$$

and (25.19)

$$e^{\omega t} = 2e^{-\nu m^2\left(1 - \frac{\ell_1^2}{m^2} + \frac{\ell_2^2}{m^2}\right)t} \cos 2\frac{\ell_1\ell_2}{m^2}t .$$

One may establish that $1 - \ell_1^2/m^2 + \ell_2^2/m^2 > .534$, so that this motion consists of damped standing-wave oscillations. The larger m is, the more quickly it is damped.

Because of the relative complexity of equations (25.16) and (25.17), it is convenient and leads to more perspicuous results to find the relation between ω and m in the two limiting cases of small and large viscosity. First consider the case of small viscosity. If in (25.17) one lets $\nu \to 0$, one regains the relation $\omega^2 = -gm - T'm^3$ of (24.9); let $\sigma_0^2 = gm + T'm^3$. However, if one retains all terms of the first power in ν, (25.17) becomes

$$\omega^2 + 4\nu m^2\omega + gm + T'm^3 = 0, \tag{25.20}$$

which has roots

$$-2m^2\nu \pm \sqrt{4m^4\nu^2 - gm - T'm^3} \doteq -2m^2\nu \pm i\sigma_0 . \tag{25.21}$$

Hence the surface profile can be described by

$$\eta = A_0 e^{-2m^2\nu t} \cos(\sigma_0 t + \tau)\cos(mx + \alpha). \tag{25.22}$$

To this order of approximation, the frequency σ_0 is related to m as in a perfect fluid, but the amplitude is gradually damped. To have some idea of the orders of magnitude involved in the damping, one should consult the table on page 402 where the row τ_0 gives computations relevant to this.

In order to find the behavior for large ν, divide equation (25.17) by $4m^4\nu^2$ and expand the term $[1+\omega/m^2\nu]^{1/2}$ in a series. If one retains only terms in ν^{-1} and ν^{-2}, the resulting equation leads to

$$3\omega^2 + 4m^2\nu\omega + 2(gm + T'\omega^3) = 0. \tag{25.23}$$

The two solutions, both real and negative, are approximately

$$\omega_1 = -\frac{gm + T'm^3}{2m^2\nu}, \quad \omega_2 = -\frac{4}{3}m^2\nu. \tag{25.24}$$

Here $|\omega_1| < |\omega_2|$ and hence ω_1 is the more important root inasmuch as it represents a slower damping of the motion. As is pointed out by Lamb [1932, p. 628], the root ω_1 corresponds to a value of ℓ only slightly less than m, so that the motion is nearly irrotational. It should also be noted that by different methods of analyzing (25.17) for large ν one may obtain somewhat different coefficients for ω_2.

In the preceding analysis it was assumed explicitly that ℓ had a non-vanishing real part. If ℓ is pure imaginary, $\ell = i\ell'$, another family of solutions exists. It is now convenient to write ϕ and ψ in the forms

$$\phi = a e^{my} e^{\omega t} \cos(mx + \alpha),$$

$$\psi = (c \cos \ell' y + d \sin \ell' y) e^{\omega t} \sin(mx + \alpha), \qquad (25.25)$$

where

$$\omega = -\nu(\ell'^2 + m^2) < 0. \qquad (25.26)$$

The motion is thus a purely subsiding one. The boundary conditions determine the following relations between a, c, and d:

$$a = c\,\frac{m^2 - \ell'^2}{2m^2}, \quad d = c\,\frac{1}{4\nu^2 m^3 \ell}\left[\nu^2(m^2 - \ell'^2)^2 + gm + T'm^3\right]. \qquad (25.27)$$

All real values of ℓ' are now admissible. The surface profile is given by

$$\eta = c\,\frac{1}{2m\nu}\, e^{\omega t} \cos(mx + \alpha). \qquad (25.28)$$

The two sets of solutions may now be used to investigate the development of an initial disturbance (cf. Sretenskii [1941]).

Forced standing waves. We may apply equations (25.16) or (25.17) to answer the following question. Suppose that m is given. Can we determine p_0 in such a way that a steady standing wave

$$\eta = A_0 e^{-i\sigma t} \cos(mx+\alpha) \tag{25.29}$$

of prescribed frequency σ is maintained? From (25.17) p_0 is then determined by

$$-m\frac{p_0}{\rho A_0} = (2m^2\nu - i\sigma)^2 - 4\nu^2 m^3\sqrt{m^2 - i\frac{\sigma}{\nu}} + gm + T'm^3. \tag{25.30}$$

If, for small viscosity, one discards terms higher than the first in ν, one obtains

$$p_0 = 4i\sigma\mu m A_0 - \sigma^2 + gm + T'm^3. \tag{25.31}$$

If we take $\sigma^2 = gm + T'm^3$, the frequency obtained from perfect-fluid theory, the necessary pressure distribution becomes

$$\bar{p} = 4\sigma\mu m A_0 i e^{-i\sigma t}\cos(mx+\alpha). \tag{25.32}$$

Thus the pressure must lead the surface displacement by a quarter of a period.

Standing waves-finite depth. If the fluid is of depth h, the analysis is similar to that above, but yields expressions of much greater complexity. The functions ϕ and Ψ may be shown to have the forms

$$\phi = \frac{1}{m}\left[d\,l\cosh m(y+h) + c\,m\sinh m(y+h)\right]e^{\omega t}\cos(mx+\alpha),$$
$$(25.33)$$
$$\Psi = \left[c\cosh m(y+h) + d\sinh m(y+h)\right]e^{\omega t}\sin(mx+\alpha),$$

where again

$$\omega = \nu\,(\ell^2 - m^2). \tag{25.34}$$

Let

$$L = \cosh \ell h, \quad L' = \sinh \ell h, \quad M = \cosh mh, \quad M' = \sinh mh.$$

Then c and d are related by

$$2m\,(c\,m\,M + d\,\ell\,M') - (\ell^2 + m^2)(c\,L + d\,L') = 0. \tag{25.35}$$

The relation between ℓ and m corresponding to (25.16) becomes

$$\nu^2(\ell^2+m^2)^2\,\frac{(\ell^2+m^2)(\ell L M - m L' M') - 2m^2\ell}{(\ell^2+m^2)(\ell L M' - m L' M)} - 4\nu^2 m^3 \ell\,\frac{2m(m M L - \ell M' L') - (\ell^2+m^2)}{2m(m M L' - \ell M' L)} +$$

$$+ g m + T' m^3 = -m\,\frac{p_0}{\rho A_0} \tag{25.36}$$

and the surface profile is

$$\eta = \frac{1}{2\nu m}(c\,L + d\,L')\,e^{\omega t}\cos(m x + \alpha) = A_0\,e^{\omega t}\cos(m x + \alpha). \tag{25.37}$$

The formulas become more perspicious in the case of small viscosity and no external pressure and exhibit the importance of the presence of the bottom. If in (25.36) one sets $p_0 = 0$ and retains only terms of order ν^0, $\nu^{1/2}$ and ν, the following equation results:

$$\omega^3 - m\sqrt{\nu}\,\tanh mh\,\omega^{5/2} + \frac{9}{2} m^2 \nu\,\omega^2 + (g m + T' m^3)\tanh mh\,\omega$$

$$-(g m + T' m^3)\,m\sqrt{\nu}\,\omega^{1/2} + \frac{1}{2}(g m + T' m^3)\,m^2 \nu \tanh mh = 0. \tag{25.38}$$

One may solve this equation by expanding ω in powers of $\nu^{\frac{1}{2}}$,

$$\omega = \omega_0 + \omega_1\sqrt{\nu} + \omega_2\nu + \ldots ,$$

substituting in (25.38) and keeping only terms in ν^0, $\nu^{1/2}$ and ν. The term independent of ν yields $\omega_0 = \pm i\sigma_0$, where σ_0 is given by (25.21) and is the frequency for an inviscid fluid. To the order of accuracy consistent with (25.38), one finds

$$\omega = \pm i\sigma_0 - (1 \pm i)\tfrac{1}{2}\, m\sqrt{2\sigma_0\nu}\,\operatorname{cosech} 2mh - 2m^2\nu\,\frac{\cosh 4mh + \cosh 2mh - 1}{\cosh 4mh - 1}. \tag{25.39}$$

The first two terms were given by Hough [1897]. The correct expression (25.39) was first given by Biesel [1949]; Hough had given $-2m^2\nu$ for the last term but he apparently made an error in calculation, for (25.39) was derived independently of Biesel's work and has also been checked by Carry [1956] (Basset's analysis [1888, p. 314] overlooks the terms in $\nu^{\frac{1}{2}}$).

The formula (25.39) should be compared with (25.21), the corresponding formula for infinite depth. There the effect of viscosity occurs only with the first power of ν. The dissipation of energy in the body of the fluid is evidently of less importance than in the vicinity of the bottom. When two fluids are superposed, a similar phenomenon occurs in the neighborhood of the interface [cf. (25.44)].

<u>Standing waves-stratified fluids</u>. Consider now the situation in which a fluid typified by ρ_1 and μ_1 fills the space $y < 0$ and another typified by $\rho_2 < \rho_1$ and μ_2 the space $y > 0$. The equations to be satisfied in the two fluids and at their interface are given in (10.3). The method of solution is analogous to that used for a

single fluid. However, separate functions ϕ_1, ψ_1, and ϕ_2, ψ_2 are needed for the lower and upper fluids. For a standing-wave solution they may be taken in the form

$$\phi_1 = a_1 e^{\omega t} e^{my} \cos(mx+\alpha) \quad , \quad \psi_1 = b_1 e^{\omega t} e^{l_1 y} \sin(mx+\alpha),$$

$$\phi_2 = a_2 e^{\omega t} e^{-my} \cos(mx+\alpha) \quad , \quad \psi_2 = b_2 e^{\omega t} e^{-l_2 y} \sin(mx+\alpha), \tag{25.40}$$

where we assume both l_1 and l_2 to have positive real parts. ω, l_1, l_2 and m are related by the equation

$$\omega = \nu_1 (l_1^2 - m^2) = \nu_2 (l_2^2 - m^2). \tag{25.41}$$

Substitution of (25.40) in the various boundary conditions at $y=0$ gives four homogeneous equations relating a_1, a_2, b_1, and b_2. The determinant of the coefficients set equal to zero yields another relation between ω, l_1, l_2 and m:

$$\left[(\rho_1+\rho_2)\omega^2 + (\rho_1-\rho_2) g m + T m^3\right]\left[\mu_1 m + \mu_2 l_2 + \mu_2 m + \mu_1 l_1\right] + 4\omega m(\mu_1 m + \mu_2 l_2)(\mu_2 m + \mu_1 l_1) = 0. \tag{25.42}$$

In the limiting case of small viscosity, (25.42) gives

$$\omega^2 + \frac{4m}{\rho_1+\rho_2} \frac{\sqrt{\rho_1\rho_2\mu_1\mu_2}}{\sqrt{\mu_1\rho_1}+\sqrt{\mu_2\rho_2}}\, \omega^{3/2} + \frac{(\rho_1-\rho_2) g m + T m^3}{\rho_1+\rho_2} = 0. \tag{25.43}$$

This has the approximate solutions, when the coefficient of $\omega^{3/2}$ is small relative to the last term,

$$\omega = \pm i\sigma_0 - \frac{1\pm i}{\sqrt{2}} \sqrt{\sigma_0}\, \frac{2m}{\rho_1+\rho_2} \frac{\sqrt{\rho_1\rho_2\mu_1\mu_2}}{\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2}} - \frac{2m^2}{\rho_1\rho_2} \frac{\rho_1\mu_1^2 + \rho_2\mu_2^2}{(\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2})^2}, \tag{25.44}$$

where σ_0 is the perfect-fluid frequency given in equation (24.14). This solution was first given by Harrison [1908]. The most significant physical fact about (25.44) when compared with (25.21) is that, to the order of approximation considered, the latter shows a rate of decay proportional to $m^2\nu$ and no influence of viscosity on the frequency, whereas (25.44) shows a rate of decay and an alteration of the frequency proportional to $m\sqrt{\nu}$ (in a dimensional sense). The greater importance of viscosity for stratified fluids may be ascribed to the different boundary condition at the interface. Harrison computed the wave velocity and modulus of decay (time required for the amplitude to decrease by a factor e^{-1}) for an air-water interface at 17°C ($\rho_1 = 1$, $\rho_2 = 0.00129$, $\nu_1 = .0109$, $\nu_2 = 0.139$, $T = 74$ in c.g.s. units). In the following table reproduced from Harrison's paper v_0, v_c and v are the wave velocities neglecting, respectively, both surface tension and viscosity, viscosity, and neither; τ_0, τ, τ_c are the moduli of decay taking account of the water viscosity only, a water-air interface without surface tension and a water-air interface with surface tension.

Wave-length (cm)	1	10	100	1000
v_0 (cm/sec)	12.48	39.46	124.79	394.62
v_c	24.90	40.05	124.81	394.62
v	24.89	40.04	124.81	394.62
τ_0	1.162"	1'56.2"	3 hrs 12'39.4"	321 hrs 5'40"
τ	1.125"	1'34.1"	1 hr 21'40.6"	36 hrs 50'36"
τ_c	1.106"	1'34.0"	1 hr 21'40.3"	36 hrs 50'34"

A striking aspect is the apparent importance of the air-water interface in damping long waves and almost total lack of influence on wave velocity (the latter fact is obvious from (25.44)).

For very large viscosities the results are analogous to those for a single fluid. The two values of ω analogous to those in (25.24) are

$$\omega_1 = -\frac{(\rho_1-\rho_2)gm+Tm^3}{\rho_1+\rho_2}\,\frac{1}{2m^2}\,\frac{\rho_1+\rho_2}{\mu_1+\mu_2}\,,\quad \omega_2 = -m^2\,\frac{\mu_1+\mu_2}{\rho_1+\rho_2}\,. \tag{25.45}$$

The analysis of the roots of (25.42) for general values of ν_1 and ν_2 is difficult. However, it has been carried through by Chandrasekhar [1955, especially pp. 170-173] for the special situation $\nu_1 = \nu_2$ and $T=0$. In this case $\ell_1 = \ell_2$. The behavior is similar to that described for (25.17) except that the critical value K_c separating a steadily decaying motion from an oscillatory decaying one is now a function of $(\rho_1-\rho_2)/(\rho_1+\rho_2)$. This value (actually a different one since he chooses a different parameter) is tabulated for a variety of density combinations. Further analysis of (25.17) may be found in a paper by Hide [1955] and Tchen [1956b].

Kusakov [1944] has carried through an analysis similar to Harrison's when the upper fluid is of depth h_2, the lower of depth h_1. However, the results do not seem to be consistent with Harrison's (or those above) when h_1 and h_2 become large. This problem has also been considered by Hide [1955], but with an approximation that neglects the viscous boundary conditions on the walls. Harrison, in the same paper, has treated also the problem when the upper fluid is of finite depth and with a free surface. We shall

not reproduce the results except to remark that his computations show a very marked influence on the damping of a thin layer of fluid of slightly different density. The effect of a variable surface tension upon wave motion is investigated briefly in Lamb [1932, § 351] and at some length in Levich [1952, pp. 477-490].

Pulsing stationary source. Dmitriev [1953] has derived the form of the functions ϕ and ψ and the surface profile in the presence of a submerged source of pulsating intensity $-Q\cos\sigma t$. We shall give here only his expression for the surface profile and an asymptotic expression for large distances from the source. Let the source be located at $(0, -h_0)$ and let

$$h = h_0\sqrt{\frac{\sigma}{\nu}}\,, \quad \bar{x} = x\sqrt{\frac{\sigma}{\nu}}\,, \quad \bar{y} = y\sqrt{\frac{\sigma}{\nu}}\,, \quad \varepsilon = \frac{\sigma^2}{g}\sqrt{\frac{\nu}{\sigma}}\,.$$

The surface profile is then represented by

$$\eta = \mathrm{Re}\,\frac{Qe^{i\sigma t}}{\pi}\,\frac{\sigma}{g}\int_0^\infty \frac{1-2iX^2}{4\varepsilon\left[iX^3\left(X-(i+X^2)^{1/2}\right)-X^4\right]+i(X-\varepsilon)}\,e^{-hX}\cos\bar{x}X\,dX \tag{25.46}$$

$$= Q\,\frac{\sigma}{g}\left(1+100\,\varepsilon^4\right)^{1/2} e^{-h\varepsilon-4\varepsilon^3\bar{x}}\cos\left(\sigma t-\varepsilon\bar{x}+4\varepsilon^3 h-\arctan 10\,\varepsilon^2\right)+\cdots.$$

26. Stability of free surfaces and interfaces.

In this section we wish to examine the circumstances under which a small disturbance of a free surface or of an interface between two fluids will increase in magnitude with time. The energy for this increase may come either from available potential energy, e.g. if the lower fluid is lighter than the upper one, available kinetic energy in the case of flowing fluids, from forced motion

of solid boundaries, or possibly some other source such as a given pressure distribution over a free surface. Surface tension and viscosity may be expected to have a stabilizing effect, so that special interest attaches to the study of their influence. We shall use the nature of the energy source as a convenient one for separating classes of problems, even though not every situation falls clearly into one of them.

Since the boundary conditions and equations which we shall use for the mathematical analysis have been linearized, following the assumption that the disturbances are small, one cannot expect the predictions of the theory to be valid quantitatively much beyond the initiation of an unstable motion. However, a great advantage in the use of linearized theory is that an arbitrary initial disturbance can be analyzed into Fourier components and the behavior of individual components examined separately.

26α. Interface between stationary superposed fluids.

Following our earlier notation, let us identify quantities referring to the lower fluid by the subscript 1 and to the upper fluid by 2. Let a sinusoid distrubance of wave number m exist at the interface. Consider first the case of perfect fluids with no surface tension. Then, if both fluids are infinitely deep, the relation (14.28) must hold. If $\rho_1 > \rho_2$, the standing-wave solution of section 14δ obtains. However, if $\rho_1 < \rho_2$, then $\sigma^2 < 0$ and σ is imaginary. Let $\omega^2 = -\sigma^2$, i.e. $\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$. Then one must replace $\cos(\sigma t + \tau)$ in the ϕ_i of that section by, say, $\sinh \omega t$. The profile of the free surface is then, according to

(10.8), given by

$$\eta = A \sin m x \cosh \omega t. \tag{26.2}$$

The amplitude of the initial corrugations of the surface evidently increases very rapidly with time, and the solution is a valid approximation for only a limited time interval. The nature of the disturbance need not have been restricted to $\sin mx$; any function $\varphi(x, z)$ satisfying $\Delta\varphi + m^2\varphi = 0$ would have yielded the same behavior. Equation (26.1) still holds if the two fluids are bounded below and above, respectively, by $y = -h_1$ and $y = h_2$ except that ω is given by

$$\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 \coth m h_2 + \rho_1 \coth m h_1} < \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}. \tag{26.3}$$

The surface is still unstable, but the rate of growth of the amplitude is slower.

Effect of surface tension. Let us now suppose that surface tension acts at the interface. Then the relation between σ and m given in (24.14) or (24.15) must hold, and a standing-wave solution is possible even if $\rho_2 > \rho_1$, provided that (24.16) holds, i.e.

$$\rho_2 < \rho_1 + \frac{T m^2}{g}. \tag{26.4}$$

Thus the interface is stable under small disturbances of sufficiently small wave length. If the unequality in (26.4) is reversed and we again set $\omega^2 = -\sigma^2$, then (26.2) holds once more and the solution is unstable. However, the value of ω^2 is less than that when $T = 0$, so that the rate of growth of the disturbance is retarded. It is also clear from the form of the relationship between ω^2 and m

that there is a wave number for which ω^2, that is the rate of growth of the disturbance, is a maximum. If both fluids are of infinite depth this mode of maximum instability occurs when

$$m^2 = (\rho_2 - \rho_1) g / 3T. \tag{26.5}$$

The effect of finite depth of the fluids is to displace the position of the maximum to higher values of m (smaller wave lengths) but a precise calculation requires solving a transcendental equation.

Effect of viscosity. The influence of viscosity in stabilizing interfacial disturbances has been the subject of a number of recent papers, in particular Bellman and Pennington [1954], Chandrasekhar [1955], Hide [1955] and Tchen [1956]. The relevant equation relating ω and m is now (25.42). Because of the high degree of this equation it is not easy to give a complete discussion of its admissible roots. However, it is easy to establish that if

$$(\rho_1 - \rho_2) g + T m^2 < 0, \tag{26.7}$$

then (25.42) has a positive real root ω_0 satisfying

$$0 < \omega_0 < \sqrt{(\rho_2 - \rho_1) g m - T m^3} \tag{26.8}$$

Thus the presence of viscosity does not alter the conditions for instability, as the presence of surface tension did, but it does have a stabilizing effect in that the rate of growth of a disturbance is slower.

In order to show the existence of a positive root under condition (26.7), one can write (25.42) in the form

$$(\rho_1+\rho_2)\,\omega^2+(\rho_1-\rho_2)\,g\,m+T m^3=-4\,\omega\, m\,\frac{(\mu_1 m+\mu_2 l_2)(\mu_2 m+\mu_1 l_1)}{\mu_1 m+\mu_1 l_2+\mu_2 m+\mu_2 l_1}$$

and sketch as functions of ω the curves represented by the two sides of the equation (remembering that l_1 and l_2 are functions of ω). The statement above then follows easily from the fact that both curves are continuous and the one represented by the right-hand function starts at the origin like

$$-2m^2(\mu_1+\mu_2)\,\omega$$

and goes to $-\infty$ in the fourth quadrant, behaving as $\omega\to\infty$ like

$$-4\,\omega^{3/2}\, m\,\frac{\sqrt{\rho_1\rho_2\,\mu_1\mu_2}}{\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2}}\ .$$

A more elaborate discussion of the roots is given by Bellman and Pennington [1954].

The behavior of ω_0 as a function of m in the interval defined by (26.7) and in particular the mode of maximum instability has been investigated by the authors cited earlier. Chandrasekhar has computed the curves $\omega_0(m)$ for $\nu_1=\nu_2$, $T=0$ and a number of values of $(\rho_2-\rho_1)/(\rho_2+\rho_1)$. Hide has recomputed these by an approximate method and then applied the method further to a fluid of finite depth with a continuous density variation $\rho_0 e^{\beta z}$. Tchen has devised a different method of approximate computation and includes the effect of surface tension. Figure 33, which is chiefly qualitative, shows the variation of ω^2 as a function of m in the interval of instability.

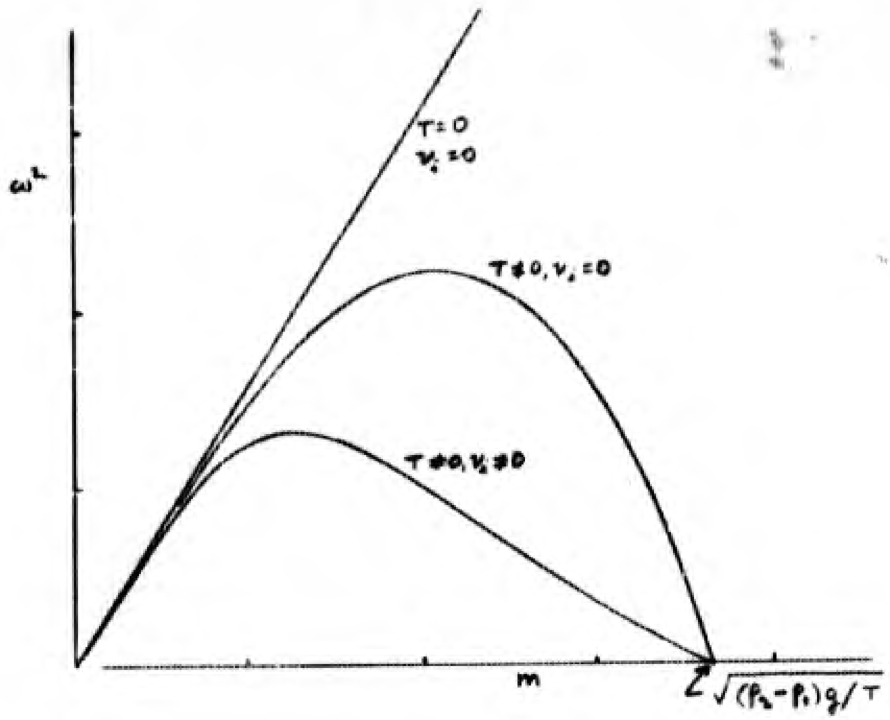


FIGURE 33

Accelerated fluid. If the whole system of fluid is being accelerated in the y-direction by a constant amount $\dot{v}_0 = g_1$, then the relative motion in a moving coordinate system is the same as if the system were at rest and g had been replaced by $g + g_1$, as is immediately evident from equation (2.15). With this change the reasoning of the preceding paragraphs still applies. This fact was pointed out by G. I. Taylor [1950] who, on the basis of it, formulated the following rule (neglecting the influence of surface tension): If the fluids are being accelerated in a direction from the more to the less dense fluid, the interface is stable; in the converse case it is unstable. Experiments carried out by Lewis [1950] for large accelerations, about 50g, confirm Taylor's observation

and the predicted initial rate of growth. Taylor's paper gave rise to a number of others treating various aspects of the instability of accelerated interfaces. In addition to those cited in the last paragraph, we mention Ingraham [1954], Plesset [1954], Birkhoff [1954], and Layzer [1955] but shall not summarize the contents. The effect of an imposed acceleration oscillating in magnitude will be discussed in section 26 γ.

26 β. Interface between moving fluids.

Consider the situation in which the fluid occupying the region $y<0$ $(y>0)$ is moving to the left with velocity $-c_1$ $(-c_2)$, and suppose that a small disturbance exists near the interface. If we suppose that the fluid is perfect and the motion in each fluid irrotational, then we may describe it by the velocity potentials

$$\Phi_i(x,y,z,t) = -c_i x + \phi_i(x,y,z,t). \tag{26.9}$$

We shall assume $c_1 \neq c_2$.

The kinematic boundary condition at the interface may be written, after linearization appropriate to the assumption of a small disturbance, in the form:

$$\eta_t(x,y,t) = c_1 \eta_x + \phi_{1y}(x,0,z,t) = c_2 \eta_x + \phi_{2y}(x,0,z,t). \tag{26.10}$$

The dynamical boundary condition (3.9) yields the following generalization of (10.8):

$$\rho_1(\phi_{1t} - c_1\phi_{1x}) - \rho_2(\phi_{2t} - c_2\phi_{2x}) + (\rho_1-\rho_2)g\eta = T(\eta_{xx}+\eta_{zz}) \text{ for } y=0. \tag{26.11}$$

If η is eliminated between (26.10) and (26.11), one finds

$$\rho_1(\phi_{1tx} - c_1\phi_{1xx}) - \rho_2(\phi_{2tx} - c_2\phi_{2xx}) + \frac{\rho_1 - \rho_2}{c_1 - c_2} g(\phi_{2y} - \phi_{1y}) \tag{26.12}$$
$$+ \frac{1}{c_1 - c_2} T(\phi_{2yyy} - \phi_{1yyy}) = 0.$$

Let us now restrict our attention to two-dimensional motion of fluids bounded above by $y = h_2$ and below by $y = -h_1$, and let the initial displacement be $\eta(x,0)$. Then from (15.2) we know that the subsequent motion may be resolved into harmonic progressive waves moving to the right and left. It will be sufficient for our purpose to examine a single component of the spectrum. Hence, we look for a solution in the form

$$\phi_1 = a_1 \cosh m(y + h_1)\, e^{i(mx - \sigma t)},$$
$$\phi_2 = a_2 \cosh m(y - h_2)\, e^{i(mx - \sigma t)}. \tag{26.13}$$

It follows from (26.10) that $(c_1 - c_2)\eta_x = -\phi_{1y} + \phi_{2y}$. Hence

$$\eta = \frac{-i}{c_1 - c_2}\left[a_1 \sinh m h_1 + a_2 \sinh m h_2\right] e^{i(mx - \sigma t)}. \tag{26.14}$$

It then follows from (26.10) that

$$\frac{a_1 \sinh m h_1 + a_2 \sinh m h_2}{c_1 - c_2} = \frac{a_1 m}{\sigma + c_1 m} \sinh m h_1 = -\frac{a_2 m}{\sigma + c_2 m} \sinh m h_2. \tag{27.15}$$

Substitution of (26.13) in (26.12) and use of (26.15) yields the following relation between σ and m :

$$\rho_1(\sigma + c_1 m)^2 \coth m h_1 + \rho_2(\sigma + c_2 m)^2 \coth m h_2 - (\rho_1 - \rho_2) g m - T m^3 = 0. \tag{26.16}$$

The solution may be expressed as follows:

$$\frac{\sigma}{m} = - \frac{c_1 \rho_1 \coth m h_1 + c_2 \rho_2 \coth m h_2}{\rho_1 \coth m h_1 + \rho_2 \coth m h_2} \pm$$

$$\pm \sqrt{\frac{(\rho_1 - \rho_2)\frac{g}{m} + T m}{\rho_1 \coth m h_1 + \rho_2 \coth m h_2} - (c_1 - c_2)^2 \frac{\rho_1 \rho_2 \coth m h_1 \coth m h_2}{(\rho_1 \coth m h_1 + \rho_2 \coth m h_2)^2}} \tag{26.16}$$

It is evident from the form of the term under the radical that σ cannot be real unless

$$(\rho_1 - \rho_2)\frac{g}{m} + T m > (c_1 - c_2)^2 \; \frac{\rho_1 \rho_2 \coth m h_1 \coth m h_2}{\rho_1 \coth m h_1 + \rho_2 \coth m h_2} . \tag{26.17}$$

It is thus evident that there are no real solutions unless the left-hand side is positive and that there may even then exist an interval of wave numbers for which the disturbances is unstable (if both g and T are zero, such a velocity discontinuity is always unstable). If one assumes $\rho_1 > \rho_2$, the minimum value of the left-hand side is

$$2 \sqrt{(\rho_1 - \rho_2) g T} \tag{26.18}$$

and occurs for $m^2 = (\rho_1 - \rho_2) g / T$. Since

$$\frac{\rho_1 \rho_2 \coth m h_1 \coth m h_2}{\rho_1 \coth m h_1 + \rho_2 \coth m h_2} > \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} , \tag{26.19}$$

the disturbance will be unstable for some wave numbers whenever

$$(c_1 - c_2)^2 > 2\,\frac{\rho_1+\rho_2}{\rho_1-\rho_2}\sqrt{(\rho_1-\rho_2)gT}\,. \tag{26.20}$$

One may conclude from (26.19) that the horizontal walls have a destabilizing effect in the sense that wave numbers which are stable for infinitely deep fluids may become unstable modes in the presence of walls. For an air-water interface the right side of (26.20) is about (646 cm/sec)^2. The corresponding wave length is 1.71 cm; if the water is at rest $(c_1 = 0)$, then the wave velocity is 0.84 cm/sec in the direction of the wind.

Let us suppose that c_1 and c_2 are both positive, i.e. that both fluids really do flow to the left. Then it follows from (26.16) that, if the roots are real, one of them is always negative and thus, from (26.13), represents a wave moving along the interface in the direction of the stream. The other will propagate upstream if

$$(\rho_1 - \rho_2)\frac{g}{m} + Tm > \rho_1 c_1^2 \coth mh_1 + \rho_2 c_2^2 \coth mh_2, \tag{26.21}$$

otherwise also downstream.

An investigation along the above lines of the stability of an interface between flowing fluids was first given by Kelvin [1871]. Similar treatments with additional information may be found in many texts, especially Lamb [1932, §§ 232, 268] and Rayleigh's <u>Theory of Sound</u> [Cambridge, 1929, § 365]. Kelvin's intention was to try to predict the minimum wind velocity which will cause a small disturbance on smooth water to increase in amplitude, and to find the unstable wave lengths. The predicted minimum velocity, roughly 650 cm/sec, is much higher than the observed minimum which is about

100 cm/sec. An evident objection to the analysis above is that viscosity of both air and water has been neglected. Since this alters in an essential way the behavior of the fluids near the interface, it is not surprizing that the prediction is not accurate. One should not expect confirmation except in circumstances in which it is possible to show that the effect of viscosity is confined to a neighborhood of the interface small with respect to the minimum wave lengths considered. The subject of wind generation of waves is still in an unsettled state. One will find summaries of the present status in the article by H. U. Roll in vol. 48 of this encyclopedia, especially pages 703-717, and also in a critical exposition by Ursell [1956]. A summary of some of the work in the USSR on wave generation is included in Shuleikin [1956].

The inclusion of viscosity in the analysis above leads to a somewhat more difficult development than in the case of standing waves. An exposition of the present achievements in this theory will be omitted; they consist chiefly of papers by Wuest [1949] and Lock [1951, 1954].

26γ. Vertically oscillated basins.

Let S denote the wetted surface of a basin and F the water surface when the basin is at rest. We shall suppose that the basin is being oscillated in the y-direction according to some given law, which may be specified by giving $v_0(t)$, the velocity of a point of the basin. It will be most convenient to describe the motion of the fluid in coordinates fixed in the basin; these will be denoted by x, y, z. We shall assume the oscillations and

and the resulting motion to be of small amplitude so that we may linearize the equations and boundary conditions.

If ϕ is the velocity potential for the motion relative to the basin and η the profile of the surface, both in coordinates fixed in the basin, then it follows easily from (2.17) that the only necessary change is to replace g by $g + \dot{v}_0$ in the boundary conditions at the free surface. They become:

$$\eta_t(x, z, t) = \phi_y(x, 0, z, t), \tag{26.22}$$

$$(g + \dot{v}_0)\eta + \phi_t(x, 0, z, t) = T'(\eta_{xx} + \eta_{zz}), \quad T' = T/\rho. \tag{26.23}$$

On the basin walls one must have

$$\phi_n = 0 \quad \text{on } S. \tag{26.24}$$

We wish, as usual, to investigate the character of the motion of the fluid.

The problem formulated above is clearly related to the problem considered in section 23γ. However, the resulting motions are quite different. Rayleigh [1883] appears to have made the first theoretical investigation of this problem. More recently it has been studied by Moiseev [1953, 1954], Benjamin and Ursell [1954], Schultz-Grunow [1955] and Bolotin [1956]. Moiseev's analysis is the most general in that the only restriction upon the basin shape is that it should allow construction of a Green's function for the Neumann problem; surface tension is not taken into account. Benjamin and Ursell restrict themselves to basins in the form of a vertical cylinder with horizontal bottom, but include the effect

of surface tension. However, at the intersection with the walls they assume a 90° angle of contact with the free surface. This is in contradiction with the observed behavior of fluids but simplifies the mathematical treatment. In spite of this shortcoming it seems desirable to include the effect of surface tension, and this will be done below. Bolotin's paper considers a modification for viscous damping. The treatment below follows closely that of Benjamin and Ursell.

Let the basin be of depth h, let C denote the intersection of the walls with the plane $y=0$, and let n be a normal to the wall at a point of C. Then, from (26.22) and (26.24) it follows that $\eta_{tn} - \phi_{yn} = 0$, or η_n = const. at each point of C; we take this constant to be zero, thus assuming a 90° contact angle with the wall. It then follows from (26.23) that $(\eta_{xx} + \eta_{zz})_n = 0$.

Let $\varphi_k(x, y, z)$ be a set of functions harmonic in the region bounded by the basin and the plane $y=0$ and satisfying (26.24), and such that $\varphi_k(x,0,z)$ form a complete set of orthonormal functions in the area of the (x,z)-plane bounded by C. Then $\phi(x,0,z,t)$, $\eta(x,z,t)$ and $\eta_{xx} + \eta_{zz}$ can each be expanded in series in $\varphi_k(x,0,z)$. The expansion of $\phi(x,0,z,t)$ determines $\phi(x,y,z,t)$ as a series in $\varphi_k(x,y,z)$. In the case at hand, when the basin is a vertical cylinder, one may separate variables as in section 12 α and construct a set φ_k in the form

$$\varphi_k(x,y,z) = \frac{\cosh m_k(y+h)\, \psi_k(x,z)}{\cosh m_k h}, \tag{26.25}$$

where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi_k(x,z) + m_k^2 \psi_k(x,z) = 0. \tag{26.26}$$

The eigenvalues m_k^2 are determined by the boundary condition on the contour C, namely $(\partial/\partial n)\varphi_k = 0$.

Let the expansion for η be written in the form

$$\eta(x,z,t) = \sum_{n=1}^{\infty} a_k(t)\,\varphi_k(x,z). \tag{26.27}$$

Then, by differentiating (26.27) and using (26.26) one gets

$$\eta_{xx} + \eta_{zz} = -\sum a_k(t)\, m_k^2\, \varphi_k(x,z). \tag{26.28}$$

If

$$\phi(x,0,z,t) = \sum b_k(t)\,\varphi_k(x,z),$$

then

$$\phi_y(x,y,z,t) = \sum b_k(t)\, m_k \frac{\sinh m_k(y+h)}{\cosh m_k h}\,\varphi_k(x,z)$$

and, from (26.22),

$$b_k(t)\, m_k \tanh m_k h = \dot{a}_k(t).$$

Hence

$$\phi(x,y,z,t) = \sum \dot{a}_k(t)\,\frac{\cosh m_k(y+h)}{m_k \sinh m_k h}\,\varphi_k(x,z). \tag{26.29}$$

Now substitute (26.27)-(26.29) in the remaining boundary condition (26.23):

$$\sum \left[(g + \dot{v}_0) a_k + T' m_k^2 a_k + \frac{1}{m_k}\ddot{a}_k \coth m_k h \right] \varphi_k = 0.$$

Since the φ_k are orthogonal, we may set each coefficient of φ_k equal to zero. With the special choice

$$\dot{v}_0 = c \cos \sigma t \tag{26.30}$$

the following set of differential equations determine the a_k :

$$\ddot{a}_k(t) + [(g m_k + T' m_k^3) \tanh m_k h + c m_k \tanh m_k h \cos \sigma t] a_k(t) = 0 \tag{26.31}$$

If we set

$$\tau = \tfrac{1}{2} \sigma t \ , \ p_k = \frac{4}{\sigma^2}(g m_k + T' m_k^3) \tanh m_k h = \frac{\sigma_k^2}{\sigma^2} \ ,$$
$$q_k = -\frac{2}{\sigma^2} c m_k \tanh m_k h, \tag{26.32}$$

where σ_k is the frequency of free oscillations in the mode m_k when the basin is fixed, then (26.31) takes one of the standard forms for the Mathieu equation:

$$\frac{d^2}{d\tau^2} a_k + [p_k - 2 q_k \cos 2\tau] a_k = 0. \tag{26.33}$$

Of particular interest in the present context is the behavior of the solutions a_k as τ, or t, becomes large. It is known from the theory of differential equations with periodic coefficients that a pair of fundamental solutions can be given in the form

$$e^{\mu\tau} Q(\tau) \ , \quad e^{-\mu\tau} Q(-\tau), \tag{26.34}$$

where Q is of period π , unless $i\mu$ is an integer. In the latter case there exists a periodic solution, of period π if $i\mu$ is even and of period 2 π if odd, and another independent nonperiodic solution. The coefficient μ will be a function of the parameters p_k , q_k and it is particularly pertinent to the present investigation

to know for what regions in the (p,q) -plane μ has a nonzero real part. These regions have been investigated for other purposes and may be found, for example, in N. W. McLachlan's Theory and application of Mathieu functions [Oxford, 1947, pp. 40,41]. In Figure 34, reproduced from Benjamin and Ursell, the shaded regions represent the unstable regions of the (p,q) -plane where τ has a nonzero real part. In the unshaded regions μ is pure imaginary (but not an integer) and the two solutions (26.34) are bounded for all τ . The

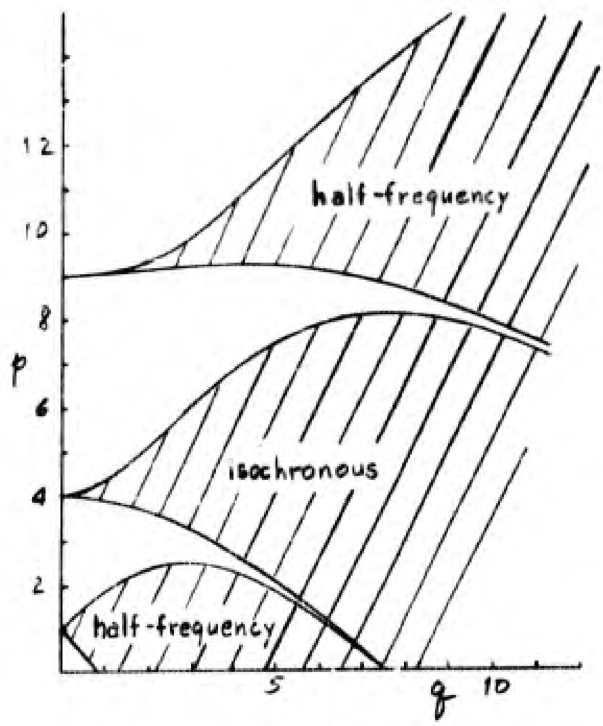


Figure 34

boundaries between regions correspond to the periodic solutions occurring when $i\mu$ is an integer. In the unstable regions the periodicity behavior of the solutions is of two types. In the

second, fourth,... regions μ is real and the solutions (26.34) are functions of period π multiplied by exponentials. In the first, third . regions $\mu = \mu_1 + i$, μ_1 real, and the solutions (26.34) now become functions of period 2 π multiplied by exponentials. In terms of t the two sets of regions correspond, respectively, to frequencies σ and $\frac{1}{2}\sigma$.

For a given mode of oscillation m_k one must compute p_k and q_k and plot (p_k, q_k) on the stability chart in order to find out whether the mode is stable or not. It seems likely, and, in fact, has been proved by Moiseev [1954, p. 44], that for any given values of σ and c some of the possible modes will be unstable. However, the analysis above has neglected the damping effect of viscosity and it may be supposed that the only unstable modes which actually occur are those associated with the smaller values of m_k. In any case, as has been emphasized earlier, the analysis is only suitable for describing the initial stages of the motion.

If the frequency of oscillation σ is equal, or nearly so, to one of the frequencies σ_k for free oscillation of the fluid, or to a subharmonic of σ_k , i.e. $\sigma = \sigma_k / n$, then $p_k = 1$, or n^2 , and it is evident from Figure 34 that (p_k, q_k) will be in an unstable region. If $\sigma = \sigma_k$, (p_k, q_k) will lie in the lowest region and standing waves with half the frequency of the basin will be generated. If $\sigma = \frac{1}{2}\sigma_k$, (p_k, q_k) will lie in the second region and the generated standing waves will have the same frequency as the basin. It is pointed out by Benjamin and Ursell that an apparent discrepancy between experimental observations of Faraday and Rayleigh and of Matthiessen can be explained by the above remarks.

Benjamin and Ursell made an experimental investigation with a circular cylinder in order to determine by experiment the boundaries of the lowest region of instability. The measurements provide a surprizingly good confirmation within certain limitations.

27. Higher-order theory of infinitesimal waves.

It is implicit in the theory of infinitesimal waves developed in the preceding sections of this chapter that the approximation given by first-order theory to the solution of a particular problem, assuming that one exists, can be improved by including further terms in the perturbation series. The solution of the resulting boundary-value problems, at least in the simplest cases, can be carried through in a manner similar to that of the first-order theory, although the computations become more and more tedious the higher the order of approximation. Nevertheless, in view of the interest of the results, the computations have been carried through by a number of persons and by a variety of methods.

Stokes [1849] was apparently the first to make the calculation for progressive waves; in fact, the method used below in section 27 α is not essentially different from Stokes' first method. Later, in connection with the publication of his collected papers, Stokes [1880] added a supplement describing a different procedure. Rayleigh turned to the problem several times [1876, 1911, 1915, 1917] and introduced still another method of approximation. It should be noted, however, that both Stokes' second method and Rayleigh's method are limited to two-dimensional irrotational progressive waves. Rayleigh [1915] seems to be the first to have given an adequate

treatment of the higher-order theory of standing waves. In addition to these classical papers there have been many others extending or improving the earlier theory; some of these will be noted below.

In all such computations, and indeed in the numerous first-order computations carried out in the earlier sections of this chapter, there is the tacit assumption that there exists an "exact solution" which is being approximated and which can be approached more amd more closely by pursuing the selected method of approximation. Unfortunately, it is seldom that one is able to prove the existence of an exact solution or of convergence of the method of approximations, and, in fact, Burnside [1916] cast doubt upon the usefulness of the Stokes-Rayleigh type of approximation of periodic progressive waves of permanent type. Burnside's objection was later met by Nekrasov's [1921, 1922, 1951], Levi-Civita's [1925] and Struik's [1926] proofs of the existence of such waves for both infinite and finite depth. However, the existence of a standing wave satisfying the exact boundary conditions has not been demonstrated as yet. The same is true of the more complicated problems considered in earlier sections. However, this mathematical shortcoming is possibly of no more importance than the neglect in many problems of relevant physical parameters such as viscosity.

One should bear in mind that the higher-order infinitesimal waves considered below are not the only higher-order approximations. The solitary and cnoidal waves of the next chapter bear a similar relation to the first-order shallow-water theory. In addition, in the last chapter another method of approximating exact waves, due to Havelock [1918], will be described.

27 α. Periodic progressive waves.

In the following we shall be seeking a wave which moves without change of form, i.e. a progressive wave in the sense of section 7. Hence we shall expect to be able to represent ϕ and η in the form

$$\phi(x,y,z,t) = \varphi(x-ct, y, z), \quad y = \eta(x-ct, z), \tag{27.1}$$

where c is the velocity of the wave. It will be convenient to represent the motion in a motion in a moving coordinate system, say $\bar{x} = x - ct$. However, we shall henceforth drop the bar over the x. The boundary conditions at the free surface are then the following:

$$\eta_x(x,z)\,\varphi_x(x,\eta(x,z)z) - \varphi_y + \eta_z\varphi_z - c\varphi_x = 0, \tag{27.2}$$

$$-c\varphi_x(x,\eta(x,z),z) + \tfrac{1}{2}(\text{grad}\,\varphi)^2 + g\eta - T'(R_1^{-1} + R_2^{-1}) = 0, \tag{27.3}$$

where $R_1^{-1} + R_2^{-1}$ is given by (3.5') and, as usual, $T' = T/\rho$. Surface tension is being taken into account both for the intrinsic interest of the results and because of an interesting phenomenon which occurs in the higher-order approximations. We shall suppose that the wave length $\lambda = 2\pi/m$ of the wave system has been given, so that c is still an unknown of the problem.

Let us now, as in section 10 α, assume that φ, η and c may all be expanded in a perturbation series in some parameter ε:

$$\begin{aligned}
\varphi &= \varepsilon\,\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \dots, \\
\eta &= \varepsilon\,\eta^{(1)} + \varepsilon^2\eta^{(2)} + \dots, \\
c &= c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots .
\end{aligned} \tag{27.4}$$

After substituting in (27.2) and (27.3) and collecting terms in the manner of section 10 α , one obtains the following boundary conditions which must be satisfied successively by $\varphi^{(1)}, \eta^{(1)}, c_0$: $\varphi^{(2)}, \eta^{(2)}, c_1$: $\varphi^{(3)}, \eta^{(3)}, c_2$:

$$c_0 \eta_x^{(1)} + \varphi_y^{(1)} = 0, \qquad g\eta^{(1)} - c_0 \varphi_x^{(1)} - T'(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}) = 0 \;;$$

$$c_0 \eta_x^{(2)} + \varphi_y^{(2)} = \varphi_x^{(1)} \eta_x^{(1)} + \varphi_z^{(1)} \eta_z^{(1)} - \eta^{(1)} \varphi_{yy}^{(1)} - c_1 \eta_x^{(1)} ,$$
$$g\eta^{(2)} - c_0 \varphi_x^{(2)} - T'(\eta_{xx}^{(2)} + \eta_{zz}^{(2)}) = c_1 \varphi_x^{(1)} - \tfrac{1}{2}(\mathrm{grad}\, \varphi^{(1)})^2 + c_0 \eta^{(1)} \varphi_{xy}^{(1)} \;; \tag{27.5}$$

$$c_0 \eta_x^{(3)} + \varphi_y^{(3)} = \varphi_x^{(2)} \eta_x^{(1)} + \varphi_z^{(2)} \eta_x^{(1)} - \varphi_{yy}^{(2)} \eta^{(1)} - c_2 \eta_x^{(1)} + \varphi_x^{(1)} \eta_x^{(2)} + \varphi_z^{(1)} \eta_z^{(2)}$$
$$- \varphi_{yy}^{(1)} \eta^{(2)} - c_1 \eta_x^{(2)} + \eta^{(1)} [\varphi_{xy}^{(1)} \eta_x^{(1)} + \varphi_{zy}^{(1)} \eta_z^{(1)}] - \tfrac{1}{2} \varphi_{yyy}^{(1)} \eta^{(1)2} , \tag{27.6}$$

$$g\eta^{(3)} - c_0 \varphi_x^{(3)} - T'(\eta_{xx}^{(3)} + \eta_{zz}^{(3)}) = c_2 \varphi_x^{(1)} + c_1 \varphi_x^{(2)} + c_1 \varphi_{xy}^{(1)} \eta^{(1)} + c_0 \varphi_{xy}^{(2)} \eta^{(1)}$$
$$+ c_0 \varphi_{xy}^{(1)} \eta^{(2)} + \tfrac{1}{2} c_0 \varphi_{xyy}^{(1)} \eta^{(1)2} + \mathrm{grad}\, \varphi^{(1)} \cdot \mathrm{grad}\, \varphi^{(2)} + \eta^{(1)} \mathrm{grad}\, \varphi^{(1)} \cdot \mathrm{grad}\, \varphi_y^{(1)}$$
$$- T' [\eta_{xx}^{(1)} \eta_z^{(1)2} + \eta_{zz}^{(1)} \eta_x^{(1)2} - 2\eta_{xz}^{(1)} \eta_x^{(1)} \eta_z^{(1)} - \tfrac{3}{2}(\eta_{xx}^{(1)} + \eta_{zz}^{(1)})(\eta_x^{(1)2} + \eta_z^{(1)2})] , \tag{27.7}$$

where all conditions are to be satisfied on the plane $y = 0$. It is possible, of course, to carry the approximations further, but three steps are ample to illustrate the procedures. The solution will be carried through in outline through the third order for infinite depth and through the second order for finite depth. As an expansion parameter we may take $\varepsilon = Am$, where A is a length determining the amplitude of the waves. The motion will be restricted to be two-dimensional.

<u>Infinite depth.</u> The solutions of (27.5) are already known from (13.5). We take them in the following form

$$\varphi^{(1)} = \frac{c_0}{m} e^{my} \sin mx, \quad \eta^{(1)} = \frac{1}{m} \cos mx, \quad c_0^2 m = g + m^2 T'. \tag{27.8}$$

After substitution in (27.9), one finds

$$\begin{aligned} &\varphi_y^{(2)} + c_0 \eta_x^{(2)} = c_1 \sin mx - c_0 \sin 2mx, \\ &c_0 \varphi_x^{(2)} - g\eta^{(2)} + T'\eta_{xx}^{(2)} = -c_1 c_0 \cos mx - \tfrac{1}{2} c_0^2 \cos 2mx. \end{aligned} \tag{27.9}$$

Elimination of $\eta^{(2)}$ yields

$$c_0^2 \varphi_{xx}^{(2)} + g\varphi_y^{(2)} - T'\varphi_{yxx}^{(2)} = 2c_1 c_0^2 m \sin mx - 3c_0 m^2 T' \sin 2mx \tag{27.10}$$

as the boundary condition to be satisfied by $\varphi^{(2)}$. If $c_1 \neq 0$, one cannot find a periodic potential function satisfying (27.10). Hence we set

$$c_1 = 0, \tag{27.11}$$

A solution of Laplace's equation satisfying (27.10) with $c_1 = 0$ and vanishing as $y \to -\infty$ is easily found to be

$$\varphi^{(2)} = \frac{3}{2} \frac{c_0}{m} \frac{m^2 T'}{g - 2m^2 T'} e^{2my} \sin 2mx, \tag{27.12}$$

providing $m^2 \neq g/2T'$. The corresponding expression for $\eta^{(2)}$ is

$$\eta^{(2)} = \frac{1}{2} \frac{1}{m} \frac{g + m^2 T'}{g - 2m^2 T'} \cos 2mx. \tag{27.13}$$

One could, of course, add terms of the form given in (27.8) but with arbitrary multipliers. However, such solutions are discarded since we wish to allow only first-order terms of this form.

Two striking facts show up in (27.12) and (27.13): First, if surface tension is neglected, $\varphi^{(2)}$ vanishes and $\varphi^{(1)}$ gives the velocity potential correctly to at least the second order. The second fact is the zero in the denominator in both $\varphi^{(2)}$ and $\eta^{(2)}$, which shows that $\varphi^{(2)}$ and $\eta^{(2)}$ become unbounded as m approaches $\sqrt{g/2T'}$. One may argue, of course, that this simply shows that validity of the perturbation method is limited to smaller and smaller values of Am the closer one comes to $\sqrt{g/2T'}$. However, it seems also to be an indication that near $m = \sqrt{g/2T'}$ the mode represented by $\varphi^{(2)}$ is of the same order of magnitude as that represented by $\varphi^{(1)}$. That this is indeed the case is clear from an examination of the equation determining $\varphi^{(1)}$ and $\varphi^{(2)}$ when $m = \sqrt{g/2T'}$. In fact, $\varphi^{(2)}$ was not determined by (27.10) for this value of m and, furthermore, (27.8) does not give the complete solution of (27.5). The solution with which we must start in this case is

$$\varphi^{(1)} = \frac{c_0}{m}\left[e^{my}\sin mx + a\,e^{2my}\sin 2mx + b\,e^{2my}\cos 2mx\right], \tag{27.14}$$

where a and b are as yet undetermined constants. Thus these two modes of motion are of the same order for $m = \sqrt{g/2T'}$. One may now substitute (27.14) and the corresponding $\eta^{(1)}$ into (27.9). By reasoning similar to that used earlier in setting $c_1 = 0$, we now find

$$a = \pm\frac{1}{2}, \quad b = 0, \quad c_1 = \pm\frac{1}{4}c_0. \tag{27.15}$$

There are thus two possible first-order modes depending upon the sign of a. $\varphi^{(2)}$ is now a sum of terms with modes $\sin 3mx$ and $\sin 4mx$, but will not be given here. The wave profile, including

modes through $\cos 2mx$, may be written as follows:

$$\eta = A\left[\cos mx + \frac{1}{2} Am \frac{g+m^2T'}{g-2m^2T'} \cos 2mx\right], \quad m \neq \sqrt{\frac{g}{2T'}} \qquad (27.16)$$

$$\eta = A\left[\cos mx \pm \frac{1}{2} \cos 2mx\right], \quad m = \sqrt{\frac{g}{2T'}}. \qquad (27.17)$$

The two signs in the second solution corresp ıd roughly to the change of sign occurring in the first when k passes through $\sqrt{g/2T'}$. Comparison of the two cases also gives an indication of the limitations upon Am necessary in the first solution, namely,

$$|Am| < \left|\frac{g-2m^2T'}{g+m^2T}\right|. \qquad (27.18)$$

A reversal of curvature at the center of the wave trough for $m < \sqrt{g/2T'}$, or of the crest for $m > \sqrt{g/2T'}$, will occur when

$$|Am| > \frac{1}{2}\left|\frac{g-2m^2T'}{g+m^2T'}\right|. \qquad (27.19)$$

The existence of the singularity in the expressions for $\eta^{(2)}$ and $\varphi^{(2)}$ was first noticed by Harrison [1909]. Wilton [1915] examined the matter more carefully, found the solutions (27.17) and, in fact, carried all approximations further. Some of Wilton's computed profiles are shown ın Figure 35. Although Wilton casts doubt upon the existence of the solution (27.17) with $+\frac{1}{2}$, such profiles seen to have been observed by Kamesvara Rav [1920]. However, the matter apparently still awaits a thorough experimental

investigation, do also similar higher modes mentioned below.

Let us now turn to the next order, assuming $m \neq \sqrt{g/2T'}$. Substitution of (27.8) and (27.11)-(27.13) into (27.7) and elimination of $\eta^{(3)}$ yields the following boundary condition to be satisfied by $\varphi^{(3)}$ on $y = 0$:

$$c_0^2 \varphi^{(3)}_{xx} + g \varphi^{(3)}_y - T' \varphi^{(3)}_{yxx} = c_0^2 m \left[2 c_2 - \frac{1}{2} c_0 \frac{2g - m^2 T'}{g - 2m^2 T'} + \frac{3}{8} c_0 \frac{m^2 T'}{g + m^2 T'} \right] \sin mx$$
$$+ \frac{9}{8} c_0^3 m \left[\frac{4 m^2 T'}{g - 2m^2 T'} - \frac{m^2 T'}{g + m^2 T'} \right] \sin 3mx . \quad (27.20)$$

Again in order to avoid an unbounded solution we must set the coefficient of $\sin mx$ equal to zero. This yields a value for c_2 :

$$c_2 = \frac{1}{2} c_0 \left[1 + \frac{3 m^2 T'}{g - 2m^2 T'} - \frac{3}{8} \frac{m^2 T'}{g + m^2 T'} \right] . \quad (27.21)$$

One may now find a potential function satisfying (27.20) and vanishing as $y \rightarrow -\infty$. The solutions for $\varphi^{(3)}$ and $\eta^{(3)}$ are as follows:

$$\varphi^{(3)} = -\frac{9}{16} \frac{c_0}{m} \frac{m^2 T'(g + 2m^2 T')}{(g - 2m^2 T')(g - 3m^2 T')} e^{3my} \sin 3mx ; \quad (27.22)$$

$$\eta^{(3)} = \frac{1}{m} \left[\frac{1}{8} + \frac{3}{16} \frac{m^2 T'(5g + 2m^2 T')}{(g + m^2 T')(g - 2m^2 T')} \right] \cos mx$$
$$+ \frac{3}{16} \frac{1}{m} \frac{2g^2 - gT'm^2 - 30(m^2 T')^2}{(g - 2m^2 T')(g - 3m^2 T')} \cos 3mx , \quad (27.23)$$

for $m \neq \sqrt{g/2T'}, \sqrt{g/3T'}$. From (27.22) one sees again that $\varphi^{(3)}$ would vanish if surface tension were neglected. Although we shall not carry through the computation, this does not happen for $\varphi^{(4)}$.

It is also evident that another singularity has appeared at $m = \sqrt{g/3T'}$. In fact, when one examines the reason for the appearance of the singularities, it is evident that a mode of the form $\cos nmx$ will always show a singularity at $m = \sqrt{g/nT'}$. In each suchcase the reason is the same as in the situation discussed earlier with $n=2$: for $m = \sqrt{g/nT'} \equiv m_n$ the proper first-order solution is of the form

$$\varphi^{(1)} = \frac{c_0}{m}\left[e^{my}\sin mx + a_n e^{nmy}\sin mx\right],$$

with a_n to be determined subsequently (according to Wilton only a_2 is not unique). Thus (27.8) should be qualified by $m^2 \neq g/nT'$. One should note that, although m_n is getting small (and hence λ_n large) as n increases, the wave number of the second first-order mode is $\sqrt{n g/T'}$. Hence, on the basis of the results in section 25, one will expect this mode to be quickly damped for large values of n. However, one may presume the first few to be observable. We remark that these special associated pairs of first-order waves always straddle the wave number for minimum c_0, namely m_1.

The wave profile, velocity potential and wave velocity will be given by

$$\eta = Am\eta^{(1)} + A^2m^2\eta^{(2)} + A^3m^3\eta^{(3)} + \cdots,$$

$$\varphi = Am\varphi^{(1)} + A^2m^2\varphi^{(2)} + A^3m^3\varphi^{(3)} + \cdots,$$

$$c = c_0 + Amc_1 + A^2m^2c_2 + \cdots. \tag{27.24}$$

To the third order the profile for pure gravity waves $(T'=0)$ is represented by the following function:

$$\eta = A\left\{\left[1+\tfrac{1}{8}A^2m^2\right]\cos mx + \tfrac{1}{2}Am\cos 2mx + \tfrac{3}{8}A^2m^2\cos 3mx + \dots\right\}$$

$$= A'\left\{\cos mx + \tfrac{1}{2}A'm\cos 2mx + \tfrac{3}{8}A'^2m^2\cos 3x + \dots\right\}, \quad (27.25)$$

where $A' = A\left[1+\frac{1}{8}A^2m^2\right]$: the velocity becomes

$$c = \sqrt{\frac{g}{m}}\left(1+\tfrac{1}{2}A^2m^2+\dots\right). \quad (27.26)$$

The velocity potential to the third order is

$$\varphi = A\sqrt{\frac{g}{m}}\, e^{my}\sin mx. \quad (27.27)$$

If one sets $g = 0$, then the wave profile for pure capillary waves becomes

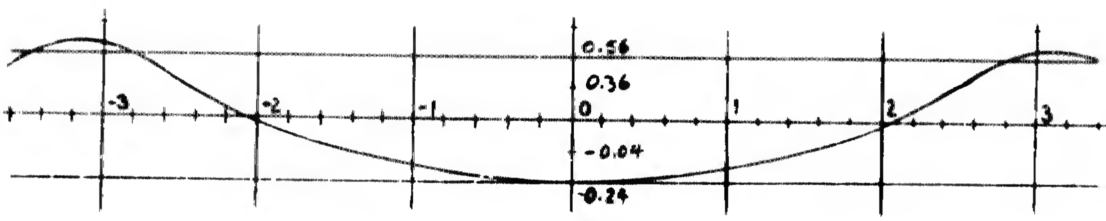
$$\eta = A\left\{\left[1-\tfrac{1}{16}A^2m^2\right]\cos mx - \tfrac{1}{4}Am\cos 2mx - \tfrac{15}{16}A^2m^2\cos 3mx + \dots\right\} \quad (27.28)$$

and the velocity

$$c = \sqrt{T'm}\left[1-\frac{7}{16}A^2m^2+\dots\right]. \quad (27.29)$$

For pure gravity waves the approximations were carried to the fifth order by Stokes, Rayleigh [1917] and others.

It is of interest to compare the profiles represented in (27.25) and (27.28). The effect of including higher-order terms in pure gravity waves is to sharpen and raise the crests and to broaden and raise the troughs. For pure capillary waves the effect is just the reverse. For combined gravity-capillary waves the increasing im-



Deep-water gravity wave : Am = .86

λ = 0.54 cm A = 0.27 cm , c = 25.7 cm/sec
(scale twice that of profiles below)

λ = 1.22 cm , A = 0.37 cm , c = 20.3 cm/sec

λ = 1.31 cm , A = 0.14 cm , c = 19.4 cm/sec

λ = 2.44 cm , A = 0.182 cm c = 24.6 cm/sec

λ = 2.44 cm , A = 0.175 cm , c = 22.2 cm/sec

Gravity-capillary waves

Figure 35

portance of the second-order term near $m = \sqrt{g/2T'}$ will first show up as a reversal of curvature at the middle of the flattened part of the wave; formula (27.19) gives the condition for the first occurrence. In Figure 35 are shown a pure gravity wave as computed by Wilton [1914] for $Am = .86$ (here A is the amplitude), and five gravity-capillary waves, the last two corresponding to the solutions (27.17), also computed by Wilton [1915] for a value of $T/\rho g = .075$. It should be remarked that the value of $Am = .86$ is much larger than any for which it is possible to prove convergence of the perturbation series and is, in fact, very close to the value of Am for the highest possible irrotational wave of permanent type (see section 33a), namely 0.891.

Finite depth. When a solid bottom is present at $y = -h$, the only necessary modification of the preceding analysis is substitution of the boundary condition $\varphi_y^{(i)}(x,-h) = 0$ for $\varphi_y^{(i)} \to 0$ as $y \to -\infty$. This increases the computational labor by a substantial amount, but otherwise introduces no difficulties. However, we call attention to the remarks on the definition of wave velocity in section 7: the velocity c below is the one defined there also a c.

The wave profile, velocity potential and wave velocity, including the effect of surface tension, are as follows, to the second order:

$$\eta = A\left\{\cos mx + \frac{1}{2}Am\frac{(2+\cosh 2mh)\operatorname{cosech} 2mh}{\tanh^2 mh - 3T'm^2(g+T'm^2)^{-1}}\cos 2mx\right\}, \tag{27.30}$$

$$\varphi = Ac_0\left\{\frac{\cosh m(y+h)}{\sinh mh}\sin mx + \frac{3}{4}Am\frac{(g+3T'm^2)\coth mh - (g+T'm^2)\tanh mh}{(g+T'm^2)\tanh^2 mh - 3T'm^2}\,\frac{\cosh 2m(y+h)}{\sinh 2mh}\sin 2mx\right\}, \tag{27.31}$$

$$c^2 = c_0^2 = \left(\frac{g}{m} + T'm\right) \tanh mh \tag{27.32}$$

The velocity is the same as in the first-order theory; this occurred also for infinite depth. In contrast to the case of infinite depth, the term $\varphi^{(2)}$ does not vanish when $T' = 0$. The singularity in the coefficient of $\cos 2mx$ still persists provided that $h > \sqrt{3T'/g}$. The earlier discussion of this phenomenon is still relevant, and a detailed one will be omitted here. However, even if surface tension is neglected in (27.30), the second-order term may still become large for small values of mh, as has been emphasized by Miche [1944]. If one again takes as an indication of increasing predominance of the second-order term a reversal of curvature at the bottom of the trough, one finds that this occurs for

$$Am > \frac{1}{2}\,\frac{\tanh^2 mh \sinh 2mh}{2 + \cosh 2mh}, \tag{27.33}$$

or approximately

$$Am > \frac{1}{3} \tanh mh \sinh^2 mh$$

as given by Miche. The occurrence of this secondary crest when mh is small has frequently been observed. It has been investigated experimentally by Morison and Crooke [1953] and by Horikawa and Wiegel [1959].

The wave profile and velocity computations were carried by Stokes to the third order for pure gravity waves in fluid of finite depth. The following expressions are taken from a report by Skjelbreia [1959]:

$$\eta = A\left\{\cos mx + \frac{1}{2} A m \frac{\cosh mh(2+\cosh 2mh)}{\sinh^3 mh} \cos 2mx + \frac{3}{16} A^2 m^2 \frac{8\cosh^6 mh + 1}{\sinh^6 mh} \cos 3mx + \dots\right\},$$

$$c^2 = \frac{g}{m} \tanh mh\left[1 + A^2 m^2 \frac{8+\cosh 4mh}{8\sinh^4 mh} + \dots\right]. \quad (27.34)$$

Skjelbreia has provided comprehensive tables allowing easy computation of η , φ and many other quantities of interest, all to the third order.

<u>Particle orbits</u>. A particularly interesting phenomenon occurs when higher-order approximations are used in the computation of the paths of individual particles. The equations which the coordinates of a particle must satisfy are

$$\frac{dx}{dt} = \varphi_x(x-ct, y), \quad \frac{dy}{dt} = \varphi_y(x-ct, y). \quad (27.35)$$

Since φ depends upon the parameter ε , the solutions x and y also will. We assume then that x and y may be expanded into series of the form

$$x(t) = x_0 + \varepsilon x_1(t) + \dots, \quad y(t) = y_0 + \varepsilon y_1(t) + \dots, \quad (27.36)$$

substitute them into (27.35) together with the appropriate expansion of φ in powers of ε , and then equate the several powers of ε separately. This results in a sequence of equations of which the first two are as follows

$$\frac{dx_1}{dt} = \varphi_x^{(1)}(x_0 - c_0 t, y_0), \quad \frac{dy_1}{dt} = \varphi_y^{(1)}(x_0 - c_0 t, y_0); \quad (27.37)$$

$$\frac{dx_2}{dt} = x_1(t)\,\varphi^{(1)}_{xx}(x_0 - c_0 t, y_0) + y_1\varphi^{(1)}_{xy} + \varphi^{(2)}_x ,$$

$$\frac{dy_2}{dt} = x_1(t)\,\varphi^{(1)}_{xy}(x_0 - c_0 t, y_0) + y_1\varphi^{(1)}_{yy} + \varphi^{(2)}_y . \qquad (27.38)$$

The first set, (27.37), was already solved in (14.17) and (14.18) and to the first order of approximation gave circular or elliptical orbits. The solution for higher orders is facilitated by neglecting surface tension and assuming $h = \infty$, for then $\varphi^{(2)}$ and $\varphi^{(3)}$ both vanish. From (27.8) one finds easily the orbit to the second order:

$$x(t) = x_0 - Ae^{my_0}\sin m(x_0 - c_0 t) + A^2 m^2 c_0 e^{2my_0} t,$$

$$y(t) = y_0 + Ae^{my_0}\cos m(x_0 - c_0 t). \qquad (27.39)$$

The circular orbits of first-order theory are now modified by a general drift in the direction of wave motion. The total amount of fluid transported per unit time (and width) is $\frac{1}{2}A^2 m c_0$. As the formula shows, this additional flow is concentrated chiefly near the surface.

When the depth is finite, or when surface tension is taken into account, the orbits become more complicated. Let

$$K = \frac{(g + 3T'm^2)\coth mh - (g + T'm^2)\tanh mh}{(g + T'm^2)\tanh^2 mh - 3m^2 T'} \qquad (27.40)$$

The particle orbits, accurate to the second order, are as follows:

$$x(t) = x_0 - A\,\frac{\cosh m(y_0+h)}{\sinh mh}\sin m(x_0 - c_0 t) + \tfrac{1}{2}A^2 m^2 c_0 t\,\frac{\cosh 2m(y_0+h)}{\sinh^2 mh}$$

$$+ \tfrac{1}{4}A^2 m\left[\operatorname{cosech}^2 mh - 3K\,\frac{\cosh 2m(y_0+h)}{\sinh 2mh}\right]\sin 2m(x_0 - c_0 t),$$

$$y(t) = y_0 + A\,\frac{\sinh m(y_0+h)}{\sinh mh}\cos m(x_0 - c_0 t) \tag{27.41}$$

$$+ \tfrac{3}{4}A^2 m K\,\frac{\sinh 2m(y_0+h)}{\sinh 2mh}\cos 2m(x_0 - c_0 t)$$

The mass-transport term in $x(t)$ is still present, and in fact, persists to the very bottom. The elliptical orbits of the first-order theory are now modified not only by the forward drift at all levels, but also by another superposed cyclic motion of twice the frequency. The effect of this is to make the orbits approximately epitrochoidal (neglecting for a moment the drift) with a small hump at the bottom which in extreme cases can become a cusp or a loop. This behavior has, in fact, been observed by Morison and Crooke [1953]. For capillary waves the situation is reversed and a dimple appears at the top.

The existence of mass transport will be reconsidered in the last chapter, where it will be demonstrated that it is a general consequence of irrotational motion when the exact boundary conditions are satisfied. The theoretically predicted monotonically decreasing forward drift with increasing depth is not confirmed experimentally for small values of kh, say $kh < 2$. Instead, with respect to a coordinate system moving with the mean velocity of the fluid, there is an observed forward flow near the bottom

and top and a backward flow in the middle portions. It is not surprizing that the perfect-fluid model does not give a good prediction for small kh, for the high shear rate near the bottom indicates that viscosity should not be neglected. Longuet-Higgins [1953b] has, in fact, devoted a long monograph to development of the higher-order theory of waves in a viscous fluid and finds theoretical drift curves agreeing qualitatively with observed ones. We shall not carry through the details here and refer to Longuet-Higgins' paper.

Wave energy. One of the striking facts about progressive first-order pure gravity waves is that the kinetic and potential energy per wave length are equal (see section 15β). This equal division of energy no longer holds when higher-order terms are taken into account. It is particularly easy to show this for $h=\infty$, for then we may use (27.25) and (27.27). The average potential energy in a wave length is

$$V_{av} = \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\eta} \rho g y\, dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{2}\rho g \eta^2 dx = \frac{1}{4}\rho g A^2\left[1+\frac{1}{2}A^2m^2\right] \qquad (27.42)$$

The average kinetic energy is

$$\mathcal{E}_{av} = \frac{m}{2\pi}\int_0^{2\pi/m} dx \int_0^{\infty} \frac{1}{2}\rho(\varphi_x^2+\varphi_y^2)\, dy = \frac{m}{2\pi}\int_0^{2\pi/m} \frac{1}{4}\rho A^2 c_0^2\, m\, e^{2m\eta}\, dx$$

$$= \frac{1}{4}\rho A^2 g\left[1+A^2m^2\right] \qquad (27.43)$$

Composite waves. Previously in this section we have been discussing a wave of permanent type whose prototype is the first-order progressive wave of the form $\eta = A\cos m(x-ct)$. It is natural to

inquire into the behavior of higher-order waves whose first-order prototype is composite, say

$$\eta = A_1 \cos m_1(x - c_1 t) + A_2 \cos m_2(x - c_2 t). \tag{27.44}$$

To find the corresponding second-order terms one may use equations (10.11) and (10.12); the computations are tedious but not difficult. The third order would introduce modifications of both c_1 and c_2 and lead to a much longer computation. As might be expected in analogy with the theory of sound, the second-order terms introduce waves of wave numbers $m_1 - m_2$ and $m_1 + m_2$, as well as $2m_1$ and $2m_2$. The velocity potential to the second order is given by

$$\varphi = A_1 c_1 e^{m_1 y} \sin m_1(x - c_1 t) + A_2 c_2 e^{m_2 y} \sin m_2(x - c_2 t)$$

$$+ 2A_1 A_2 \frac{m_1 m_2 (c_1 - c_2) g}{g(m_1 - m_2) - (m_1 c_1 - m_2 c_2)^2} e^{(m_1 - m_2) y} \sin[(m_1 - m_2)x - (m_1 c_1 - m_2 c_2)t]. \tag{27.45}$$

Biesel [1952] has derived formulas for a composite wave with a finite number of components and for h finite. He computes a number of quantities of interest. However, the formulas are very long and will not be reproduced here.

Three-dimensional waves. By using the full three-dimensional equations as given in (27.5) to (27.7) one may develop a higher-order theory of doubly modulated waves analogous to those considered in section 14 γ by first-order theory. This has been done by Fuchs [1952] and Sretenskii [1954] to whose papers we refer for the resulting motion.

Further references. Development of systematic methods of computation of higher-order approximations has recently attracted the

attention of several persons. Among these are Sretenskii [1952], Borgman and Chappelear [1957], Daubert [1957, 1958] in a series of notes, Jolas [1958] and Normandin [1957]. Sretenskii [1953, 1955] has investigated the higher-order theory of wave motion resulting from a moving pressure distribution and waves in a circular canal.

27 β. Standing waves.

As will be evident below, the formulation of a higher-order theory of standing waves is somewhat clumsier than that for progressive waves of permanent type. Part of the difficulty stems from the fact that one necessarily must deal with one more variable, namely t. The type of motion we are seeking will be represented by a profile $\eta(x,t)$ and a velocity potential $\phi(x,y,t)$ periodic in both x and t:

$$\eta(x+r\lambda, t+s\tau) = \eta(x,t), \quad \phi(x+r\lambda, y, t+s\tau) = \phi(x,y,t). \qquad (27.46)$$

If we fix the wave length $\lambda = 2\pi/m$, then the period $\tau = 2\pi/\sigma$ will have to be determined as one of the unknowns of the problem. In addition, we wish to have the first-order standing wave $\eta = A\cos mx \cos\sigma$ of section 14 α serve as a prototype and first-order solution of the more general problem. As a further condition, we shall suppose the motion to be symmetric with respect to a vertical line through a crest.

Rayleigh [1915] was apparently the first to consider this problem. It was later attacked in an entirely different way, using Lagrangian coordinates, by Sekerzh-Zenkovich [1947, 1951a,b,1952], who treated both two- and three-dimensional waves for infinite depth,

two-dimensional waves for finite depth, and composite waves for infinite depth. Penney and Price [1952], following approximately Rayleigh's method, carried the approximation for two-dimensional motion and $h = \infty$ to the fifth order, and to the second order for h finite and for doubly modulated standing waves. The method used below is a modification of theirs. The two-dimensional problem has recently been studied in a series of notes by Chabert d'Hières [1957, 1958]. Carry [1953] has carried to the second-order the superposition of two standing waves of the same wave length but 90° out of phase and of differing first-order amplitudes. Ingraham [1954] has carried to the second order the stability analysis of superposed two-fluid systems discussed at the beginning of section 26 α.

Since η and ϕ are periodic in both x and t, we may expand each in a double Fourier series. However, it is also necessary to bring into the form of the series some indications of orders of magnitudes of the components, and in such a way that the first-order term is of the desired sort. We assume the following expansions for an infinitely deep fluid:

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \dots,$$

$$\eta(x,t) = \sum_{n=1}^{\infty} \varepsilon^n \eta^{(n)} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{p,q=0}^{n} \left[a_{pq}^{(n)} \cos q\sigma t + b_{pq}^{(n)} \sin q\sigma t\right] \cos pmx,$$

$$\phi(x,y,t) = \sum_{n=1}^{\infty} \varepsilon^n \phi^{(n)} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{p,q=0}^{n} \left[c_{pq}^{(n)} \cos q\sigma t + d_{pq}^{(n)} \sin q\sigma t\right] e^{pmy} \cos pmx. \quad (27.47)$$

We may immediately set $d_{p0}^{(n)} = 0$, $b_{p0}^{(n)} = 0$ and with no loss of generality also $c_{00}^{(n)} = 0$. Since the mean water level has been fixed at $y = 0$, we must also have $a_{00}^{(n)} = 0$. We shall again

take $\epsilon = Am$, where A is the amplitude of the first-order term.

Substitution of (27.47) into the exact kinematic and dynamic boundary conditions,

$$\begin{aligned} &\phi_x(x,\eta,t)\,\eta_x - \phi_y + \eta_t = 0 \\ &\phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta - T'\left(R_1^{-1} + R_2^{-1}\right) = 0, \end{aligned} \tag{27.48}$$

results, as in section 10 $\propto$ and 27 $\propto$, in a series of equations for successive determination of the coefficients $a_{pq}^{(n)}, \ldots, d_{pq}^{(n)}$ and $\sigma_0, \sigma_1, \ldots$. Because of the assumed form of the solution, the equations are now always linear equations between the coefficients. The boundary conditions for $\phi^{(1)}$ and $\eta^{(1)}$, namely,

$$\begin{aligned} &\phi_y^{(1)} - \eta_t^{(1)} = 0, \\ &\phi_t^{(1)} + g\eta^{(1)} - T'\eta_{xx}^{(1)} = 0, \end{aligned} \tag{27.49}$$

yield

$$-\sigma_0 a_{01}^{(1)} \sin\sigma_0 t + \sigma_0 b_{01}^{(1)} \cos\sigma_0 t = 0,$$

$$-\sigma_0 a_{11}^{(1)} \sin\sigma_0 t + \sigma_0 b_{11}^{(1)} \cos\sigma_0 t - k\left[c_{10}^{(1)} + c_{11}^{(1)}\cos\sigma_0 t + d_{11}^{(1)}\sin\sigma_0 t\right] = 0; \tag{27.50}$$

$$g\left[a_{01}^{(1)}\cos\sigma_0 t + b_{01}^{(1)}\sin\sigma_0 t\right] + \left[-\sigma_0 c_{01}^{(1)}\sin\sigma_0 t + \sigma_0 d_{01}^{(1)}\cos\sigma_0 t\right] = 0,$$

$$(g + m^2T')\left[a_{10}^{(1)} + a_{11}^{(1)}\cos\sigma_0 t + b_{11}^{(1)}\sin\sigma_0 t\right] + \left[-\sigma_0 c_{11}^{(1)}\sin\sigma_0 t + \sigma_0 d_{11}^{(1)}\cos\sigma_0 t\right] = 0. \tag{27.51}$$

From these follow immediately

$$a_{01}^{(1)} = b_{01}^{(1)} = c_{10}^{(1)} = a_{10}^{(1)} = 0, \quad d_{11}^{(1)} = -\frac{\sigma_0}{m} a_{11}^{(1)}, \quad c_{11}^{(1)} = -\frac{\sigma_0}{m} b_{11}^{(1)}, \tag{27.52}$$

and

$$\sigma_0^2 = gm + m^3 T' \tag{27.53}$$

We shall in addition fix the phase by making the arbitrary choice

$$a_{11}^{(1)} = \frac{1}{m} \quad , \; b_{11}^{(1)} = 0 \; , \tag{27.54}$$

so that

$$\eta^{(1)} = \frac{1}{m} \cos mx \cos \sigma_0 t \; , \quad \phi^{(1)} = -\frac{\sigma_0}{m^2} \cos mx \sin \sigma_0 t \; . \tag{27.55}$$

This is a rather clumsy way to derive a first-order solution which was found much more directly earlier in section 14 α. However, it provides a caricature of the procedure necessary at each new stage of approximation. Since the higher-order approximations lead to extremely tedious calculations, they will be completely omitted and only the results given.

The profile and velocity potential through the second order are given by

$$\begin{aligned} \eta &= A \cos \sigma_0 t \cos mx + \frac{1}{4} A^2 m \frac{g + m^2 T'}{g + 4m^2 T'} \cos 2mx \\ &\quad + \frac{1}{4} A^2 m \frac{g + m^2 T'}{g - 2m^2 T'} \cos 2\sigma_0 t \cos 2mx \; , \\ \phi &= -A \frac{\sigma_0}{m} \sin \sigma_0 t \, e^{my} \cos mx + \frac{1}{4} A^2 \sigma_0 \sin 2\sigma_0 t \\ &\quad - \frac{1}{4} A^2 \sigma_0 \frac{3m^2 T'}{g - 2m^2 T'} \sin 2\sigma_0 t \, e^{2my} \cos 2my \; , \end{aligned} \tag{27.56}$$

for $m^2 \neq g/2T'$; here $\sigma_1 = 0$. If $m^2 = g/2T'$, the situation is similar to that discussed in section 27 α following (27.13). For this value of m we must start with a first-order solution of the

form:

$$\phi^{(1)} = -\frac{\sigma_0}{m^2}\left[\sin \sigma_0 t \, e^{my} \cos mx + b_1 \sin 2\sigma_0 t - b_2 \cos 2\sigma_0 t) e^{2my} \cos 2mx\right]$$

$$\eta^{(1)} = \frac{1}{m}\left[\cos \sigma_0 t \cos mx + (b_1 \cos 2\sigma_0 t + b_2 \sin 2\sigma_0 t)\cos 2mx\right] \tag{27.57}$$

The values of b_1 , b_2 and σ_1 are now determined by the second-order equations and are

$$b_1 = \pm \frac{1}{2} \; , \; b_2 = 0 \; , \; \sigma_1 = \frac{1}{8}\sigma_0 \; . \tag{27.58}$$

Thus the first-order profile for $m^2 = g/2T'$ is

$$\eta = A \cos mx \cos \sigma_0 t \pm \frac{1}{2} A \cos 2\sigma_0 t \cos 2mx . \tag{27.59}$$

The amplitude relation between the two first-order modes is the same as for progressive waves of this length.

The expression for the third-order standing wave is very clumsy if T' is retained. Also, as might be expected from analogy with the progressive wave, another apparent singularity appears for $m^2 = g/3T'$. If one sets $T' = 0$, the expressions for η and ϕ become much simpler and are as follows:

$$\eta = A \cos \sigma t \cos mx + \frac{1}{4} A^2 m \cos 2mx + \frac{1}{4} A^2 m \cos 2\sigma t \cos 2mx +$$
$$+ \frac{1}{32} A^3 m^2 \left[-2 \cos 3\sigma t \cos mx + 9 \cos \sigma t \cos 3mx + 3 \cos 3\sigma t \cos 3mx\right] ;$$

(27.60)

$$\phi = -\frac{\sigma A}{m} \sin \sigma t \, e^{my} \cos mx + \frac{1}{4} \sigma A^2 \sin 2\sigma t +$$
$$+ \frac{5}{32} \sigma m A^3 \sin 3\sigma t \, e^{my} \cos mx + \frac{3}{16} \sigma m A^3 \sin 3\sigma t \, e^{3my} \cos 3mx ;$$

$$\sigma = \sqrt{gm}\left(1 - \frac{1}{8} A^2 m^2\right) .$$

As has been mentioned earlier, the approximation has been carried to the fifth order by Penney and Price [1952]. However, it is not necessary to carry the approximation so far in order to see some important features of the motion, namely the sharpening of the crests and flattening of the troughs, the absence of any nodal points and the decrease of frequency with amplitude. One interesting feature does require carrying the approximation to at least the fourth order: this is the absence of any time during a period when the surface is completely flat. In connection with an experimental test of a predicted standing wave of greatest amplitude-length ratio by Penney and Price, G. I. Taylor [1953] has also provided an experimental verification of the correctness of the theory in an extreme case.

Orbits. The method of computation of orbits including higher-order terms is the same as that outlined at the end of section 27α and we omit a detailed exposition. For infinite depth and $T' = 0$ the orbits to the second order are given by

$$x = x_0 - Ae^{my_0} \sin mx_0 \cos \sigma_0 t,$$

$$y = y_0 + Ae^{my_0} \cos mx_0 \cos \sigma_0 t + \frac{1}{4} A^2 m e^{2my_0} \cos 2\sigma_0 t. \tag{27.61}$$

The effect of the last term in y is easily seen to be a small wiggle superposed on the first-order straight-line trajectories discussed in section 14 α, except directly beneath the crests where the trajectory is still vertical but with the midpoint somewhat above the equilibrium position.

Pressure distribution. A particularly interesting consequence of keeping second-order terms appears in the behavior of the pressure distribution. From (27.56) and Bernoulli's theorem one finds for the average pressure over a wave length

$$\overline{p-p_0} = \frac{1}{\lambda}\int_0^\lambda (p-p_0)\,dx = -\rho g y - \frac{1}{4}\rho A^2\sigma_0^2 e^{2my} + \frac{1}{4}\rho A^2\sigma_0^2 e^{2my}\cos 2\sigma_0 t$$

$$-\frac{1}{2}\rho A^2\sigma_0^2\cos 2\sigma_0 t \qquad (27.62)$$

The terms with e^{2my} as a factor drop off quickly. However, the last term is independent of y and at all depths yields a fluctuation about the hydrostatic pressure with double the frequency of the standing waves. The existence of this depth-independent fluctuation, deriving from the term ϕ_t in Bernoulli's theorem and the purely time-dependent term in ϕ, was pointed out by Miche [1944, p. 73]. The matter has been investigated more intensively by Longuet-Higgins who has extended the theory to include a more general wave motion and compressibility of the fluid. He has further applied the theory to give a plausible explanation of recorded microseisms. Kierstead [1952] has extended Longuet-Higgins' analysis to include two-fluid systems. Cooper and Longuet-Higgins [1951] have carried out laboratory experiments showing excellent agreement with the predicted pressure distribution for both progressive and standing waves.

Finite depth. Computations of the surface profile, particle orbits and other quantities for finite depth have been carried to the third order by Sekerzh-Zenkovich [1951] and Carry and Chabert d'Hières [1957]. We reproduce here the results only to the second

order (for pure gravity waves):

$$\eta = A \cos \sigma t \cos m x + \tfrac{1}{8} A^2 m \tanh m h \left[1 + \coth^2 m h - \coth^2 m h \,(3 \coth^2 m h - 1) \cos 2\sigma t\right] \cos 2 m x,$$

$$\phi = -A \frac{\sigma}{m} \frac{\cosh m(y+h)}{\sinh m h} \sin \sigma t \cos m x + \frac{1}{16} A^2 \sigma \,(3 + \coth^2 m h) \sin 2\sigma t$$

$$+ \frac{3}{8} A^2 \sigma \frac{\coth m h}{\sinh^2 m h} \frac{\cosh 2m(y+h)}{\sinh 2 m h} \sin 2\sigma t \cos 2 m x, \qquad (27.63)$$

$$\sigma^2 = \sigma_0^2 = g m \tanh m h, \quad \sigma_1 = 0 .$$

The pressure averaged over a wave length (cf. (27.62)) is

$$\overline{p - p_0} = -\rho g y + \frac{1}{8} \frac{A^2 \sigma^2}{\sinh^2 m h} \left[1 - \cosh 2m(y+h) - \left(2 \cosh 2 m h - \cosh 2m(y+h) - 1\right) \cos 2\sigma t\right]. \qquad (27.64)$$

On the bottom, $y = -h$, one finds

$$\overline{p - p_0} = \rho g h - \tfrac{1}{2} \rho A^2 \sigma^2 \cos 2\sigma t. \qquad (27.65)$$

We note that here also, as in the case of progressive waves, the importance of the second-order terms in η and ϕ increases as $m h \to 0$.

27γ. Waves in a viscous fluid.

The equations (10.2)-(10.4), used in section 25 in developing the first-order theory of waves in a viscous fluid, may be considered as the first in a sequence for the determination of higher-order approximations. Although the formulation of the equations appears to be straight forward, if laborious, the higher-order theory does not seem to have attracted many investigators.

Harrison [1909] made a second-order investigation of progressive waves and Longuet-Higgins [1953] has recently made an elaborate study of both progressive and standing waves in an attempt to explain certain observed features of mass transport velocities. We shall not attempt to summarize either paper. However, the following results, taken from Harrison, may be of interest. For the wave profile to the second order he gives the following expression when ν is small (cf. (25.22)):

$$\eta = Ae^{-2\nu m^2 t}\cos(mx-\sigma_0 t)$$
$$+A^2 e^{-4\nu m^2 t}\left[\tfrac{1}{2}m\cos 2(mx-\sigma_0 t)-m^2\left(\frac{\nu^2}{4gm}\right)^{\frac{1}{4}}\sin 2(mx-\sigma_0 t)\right], \tag{27.66}$$

where $\sigma_0^2 = gm$. The effect of viscosity, besides damping, is to make the leading side of the crest steeper than the trailing side. According to Harrison the average horizontal velocity of a particle, again for small ν, is

$$A^2\sigma_0 m\, e^{2my-4\nu m^2 t} - A^2 m^2\sqrt{\tfrac{1}{2}\sigma_0\nu}\left[(4\cos l_2 y+\sin l_2 y)e^{(m+l_1)y}+\sin 2my\right]e^{-4\nu m^2 t}$$
$$+A^2 m^3\nu\left[4e^{(m+l_1)y}\sin l_2 y+3e^{2l_1 y}\right]e^{-4\nu m^2 t}, \tag{27.67}$$

where, as in (25.19), $l = l_1 + il_2$ and $\nu(l^2-m^2) = \omega \doteq -2\nu m^2 + i\sigma_0$. This formula should be compared with $A^2m^2c_0e^{2my}$ computed from (27.39), to which it reduces when $\nu = 0$.